Classical Mechanics and Electromagnetism in Accelerator Physics

January 25, 2016

last updated January 24, 2016

Introduction

- We will look at selected topics in classical mechanics and electrodynamics and apply them to important topics in accelerator physics. The focus will be on rings, but the methodology applies more generally.
- The choice of material is somewhat subjective.
- I assume knowledge of basics of classical mechanics, electrodynamics, and the special theory of relativity.
- The course is designed to be self-contained. We will go over key derivations, at least briefly.
- You have lecture notes from Gennady Stupakov's class in 2011, extra handouts, and a textbook. We will only cover a portion of this material, and not always in order. The textbook is mostly for reference.
- Some materials are available online, `http://laser.lbl.gov/~gpenn/uspas2016`

Practical matters

- Daily schedule:

9 am — 12 pm	lectures (with a break around 10:30)
2 pm — 4 pm	solving sample problems, special topics, questions
Evenings	we will be available for questions
Fridays	will be short days

- Homework is due next day at 9 am.
- We will have 9 days of lectures and a final exam on Friday, Feb 5, 9 am — 12 pm. No homework will be assigned the day before the exam. A normal-length homework will be assigned on the first Friday, due Monday. Final grade is based on 60% homework + 40% exam.
- SI system of units is used throughout the course

Practical matters

- Mornings and Afternoons, here unless otherwise specified
- Evenings in same area as meals unless otherwise specified
- Teaching the course:

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Kilean Hwang	Grader kh20@indiana.edu

- We have a small class, feel free to ask questions during lectures. The pacing of the class can also be adjusted.
- Any comments on lectures and notes are highly appreciated.
- Using software packages (Matlab, Mathematica) for calculations is fine. There will be no computer lab.

Main themes of this course

- Classical Mechanics
  - Oscillators
  - Hamiltonial formulation of equations of motion
  - Action-angle variables
  - Dynamics in a ring
  - Distribution function and Vlasov equation
  - Special Relativity
- Electromagnetism
  - Self-fields of beams
  - Effect of environment
  - Radiation fields
  - Synchrotron radiation
  - Formation length of radiation, coherent effects, beam diagnostics

Recommended references

Books:

- Jorge V. José and Eugene J. Saletan. Classical dynamics: a contemporary approach. Cambridge University Press, 1998.
- J. D. Jackson. Classical Electrodynamics. Wiley, New York, third edition, 1999.
- Herbert Goldstein, Charles Poole, and John Safko. Classical mechanics. Edison-Wesley, 3d edition, 2000.
- Walter Greiner, Classical Mechanics: Systems of particles and Hamiltonian dynamcs, 2nd edition. Springer, 2010.
- Gerald Jay Sussman and Jack Wisdom. Structure and Interpretation of Classical Mechanics. MIT Press, 2001.

The last ref. is available online at:
`http://mitpress.mit.edu/SICM/`

Linear and Nonlinear Oscillators
(Lecture 2)

January 25, 2016

Lecture outline

A simple model of a linear oscillator forms the foundation of many physical phenomena in accelerator dynamics. A typical trajectory of a particle in an accelerator can be represented as an oscillation around a so called reference orbit.

We will start from recalling the main properties of the linear oscillator. We then look at the response of a linear oscillator to external force, and effects of varying oscillator frequency. We then proceed to an oscillator with a small nonlinearity, with a pendulum as a solvable example of a nonlinear oscillator. We finish with brief discussion of resonance in nonlinear oscillators.

Linear Oscillator

A differential equation for a linear oscillator without damping has the form

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0 \tag{2.1}$$

where $x(t)$ is the oscillating quantity, t is time and ω_0 is the oscillator frequency. For a mass on a spring, $\omega_0^2 = k/m$, where k is the spring constant.

Figure : A mass attached to a spring.

Linear Oscillator

General solution of Eq. (2.1) is characterized by the amplitude A and the phase ϕ

$$x(t) = A\cos(\omega_0 t + \phi) \tag{2.2}$$

This solution conserves the quantity $x(t)^2 + \dot{x}^2(t)/\omega_0^2$.

Linear Oscillator

Damping due to a friction force which is proportional to the velocity introduces a term with the first derivative into the differential equation

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = 0 \tag{2.3}$$

where γ is the damping constant (it has the dimension of frequency). A general solution to this equation is

$$x(t) = Ae^{-\gamma_1 t} \cos(\omega_1 t + \phi) \tag{2.4}$$

with

$$\omega_1 = \omega_0 \sqrt{1 - \frac{\gamma^2}{4\omega_0^2}} \qquad \gamma_1 = \frac{1}{2}\gamma\,. \tag{2.5}$$

If $\gamma \ll \omega_0$, then the frequency ω_1 is close to ω_0, $\omega_1 \approx \omega_0$. Quality factor $Q = \omega_0/2\gamma$.

Linear Oscillator

As an example, consider particle oscillations in the Low Energy Ring in PEP-II at SLAC. In the transverse direction, the particle executes the betatron oscillations with the frequency about 40 times larger than the revolution frequency of 136 kHz. This makes $\omega_\beta \sim 2\pi \times 5.4$ MHz. The damping time γ_1^{-1} due to the synchrotron radiation is about 60 ms, which means that $\gamma = 2/(60\mathrm{ms}) \approx 30$ Hz. We see that the ratio $\gamma/\omega_\beta \sim 10^{-6}$ is extremely small for these oscillations, and the damping can be neglected in first approximation.

Driven Oscillator

If the oscillator is driven by an external force $f(t)$ then we have

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = f(t) \tag{2.6}$$

($f(t)$ is properly normalized here). From the ODE theory we know how to write a general solution to the above equation. We will write down here the result for the case $\gamma = 0$:

$$x(t) = x_0 \cos\omega_0 t + \frac{\dot{x}_0}{\omega_0}\sin\omega_0 t + \frac{1}{\omega_0}\int_0^t \sin\omega_0(t-t')f(t')dt' \tag{2.7}$$

where x_0 and $\dot{x}_0$ are initial, at $t = 0$, coordinate and velocity of the oscillator.

Resonance

Let's assume that an oscillator is driven by a sinusoidal force, $f(t) = f_0 \cos \omega t$. A convenient way to study this problem is to use complex numbers. Instead of considering a real function $x(t)$ we will consider a complex function $\xi(t)$ such that $x(t) = \mathrm{Re}\,\xi(t)$. The equation for ξ is

$$\frac{d^2\xi}{dt^2} + \gamma \frac{d\xi}{dt} + \omega_0^2 \xi = f_0 e^{-i\omega t} \tag{2.8}$$

Let us seek solution in the form $\xi(t) = \xi_0 e^{-i\omega t}$ where ξ_0 is a complex number, $\xi_0 = |\xi_0| e^{i\phi}$. This means that the real variable x is $x(t) = \mathrm{Re}\ \xi(t) = \mathrm{Re}\ (|\xi_0| e^{-i\omega t + i\phi}) = |\xi_0| \cos(\omega t - \phi)$. We have

$$(-\omega^2 - i\omega\gamma + \omega_0^2)\xi_0 = f_0 \tag{2.9}$$

and

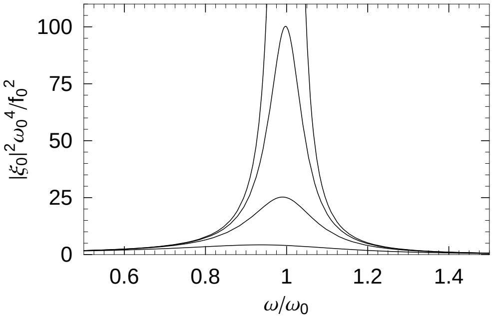
$$\xi_0 = \frac{f_0}{\omega_0^2 - \omega^2 - i\omega\gamma} \tag{2.10}$$

Resonance

For the amplitude squared of the oscillations we find

$$|\xi_0|^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2} \tag{2.11}$$

The plot of the amplitude versus frequency ω is shown below for $\gamma = 0.5,\ 0.2,\ 0.1,\ 0$



Resonance

When the damping factor γ is small we have an effect of *resonance*: the amplitude of oscillations increases when the driving frequency approaches the resonant frequency. The *width* $\Delta\omega_{\mathrm{res}}$ of the resonance is defined as a characteristic width of the resonant curve, $\Delta\omega_{\mathrm{res}} \sim \gamma$. It makes sense to talk about the resonance only when $\gamma \ll \omega_0$.

Random kicks

What happens if an oscillator is kicked at random times? Let us assume that the external force is given by the following expression,

$$f(t) = \sum_i a_i \delta(t - t_i) \tag{2.12}$$

where t_i are random moments of time, and the kick amplitudes a_i take random values. To deal with this problem we will use Eq. (2.7) (assuming for simplicity that $\gamma = 0$). We then have

$$\begin{aligned} x(t) &= \frac{1}{\omega_0} \int_0^t \sin \omega_0 (t - t') f(t') dt' \\ &= \sum_i \frac{a_i}{\omega_0} \sin \omega_0 (t - t_i) \end{aligned} \tag{2.13}$$

where we also assumed that at time $t = 0$ the oscillator was at rest. The result is a random function whose particular values are determined by the specific sequence of a_i and t_i.

Random kicks

We would like to find some statistical characteristics of this random motion. An important quantity is the sum $x(t)^2 + \dot{x}^2(t)/\omega_0^2$—for free oscillations it is equal to the square of the amplitude. So we want to find the statistical average of this quantity:

$$\begin{aligned}
&\langle x(t)^2 + \frac{\dot{x}^2(t)}{\omega_0^2}\rangle = \\
=&\langle \omega_0^{-2} \sum_{i,j} a_i a_j (\sin\omega_0(t-t_i)\sin\omega_0(t-t_j) \\
&+ \cos\omega_0(t-t_i)\cos\omega_0(t-t_j))\rangle \\
&= \langle \omega_0^{-2} \sum_{i,j} a_i a_j \cos\omega_0(t_i - t_j)\rangle
\end{aligned} \tag{2.14}$$

Because t_i and t_j are not correlated if $i \neq j$, the phase of the cosine function is random, and the terms with $i \neq j$ vanish after averaging. Only the terms with $i = j$ survive the averaging.

Random kicks

The result is

$$\left\langle x(t)^2 + \frac{\dot{x}^2(t)}{\omega_0^2} \right\rangle = \frac{\langle a^2 \rangle}{\omega_0^2} N_{\mathrm{kick}}(t) = \frac{\langle a^2 \rangle}{\omega_0^2} \frac{t}{\Delta t} \tag{2.15}$$

where $N_{\mathrm{kick}}(t)$ is the number of kicks on the interval $[0, t]$, Δt is an average time between the kicks. We see that the square of the amplitude grows linearly with time. This is a characteristic of a *diffusion* process.

Parametric resonance

Let us consider what happens if we vary parameters of our linear oscillator periodically with time. Since we have only one parameter, this means that $\omega_0(t)$ is a *periodic* function,

$$\frac{d^2x}{dt^2} + \omega_0^2(t)x = 0 \tag{2.16}$$

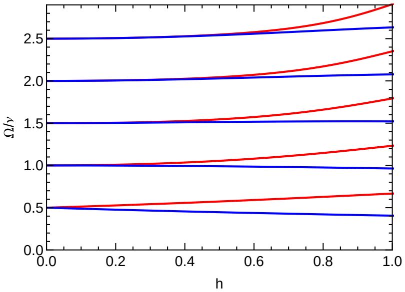
Moreover, let us assume that

$$\omega_0^2(t) = \Omega^2(1 - h\cos\nu t) \tag{2.17}$$

(the resulting equation is called the Mathieu equation). Naively, one might think that if h is small, the solution will be close to that of a linear oscillator with constant parameters. This is not always the case, as numerical solutions show. It turns out that even if h is small, oscillations become unstable if the ratio of the frequencies Ω/ν is close to $n/2$, where n is an integer. In other words, for $\nu \approx 2\Omega,\ 2\Omega/3,\ \Omega/2,\ 2\Omega/5,\ \Omega/3\ldots$ the oscillator is unstable.

Parametric resonance

The exact pattern of stable and unstable regions in the plane Ω, h is rather complicated. Unstable regions for Eq. (2.16) are bounded by red curves from above and blue curves from below.



Note that those regions become exponentially narrow if $h \lesssim 1$ and ν/Ω is small. For a small ν we have an oscillator whose parameters are varied *adiabatically* slow. Small damping makes the system stable for small h and ν.

Swing - an example of parametric resonance

Question: what is ν/Ω here?

Adiabatic variation of parameters

Consider slow variation of the parameters of the oscillator. Assume that the frequency ω_0 varies in time from a value ω_1 to ω_2 over a time interval τ:

$$\omega_0^{-2}\left|\frac{d\omega_0}{dt}\right| \ll 1 \tag{2.18}$$

The relative change of the frequency ω_0 over one radian of oscillations is small. This is an *adiabatic* regime.

How does the amplitude of the oscillations vary in time? Seek solution of Eq. (2.16) in the following (complex) form

$$\xi(t) = A(t)\exp\left(-i\int_0^t \omega_0(t')dt' + \phi_0\right) \tag{2.19}$$

where $A(t)$ is a slowly varying amplitude and ϕ is an initial phase. Put this into (2.16)

$$\frac{d^2A}{dt^2} - 2i\omega_0\frac{dA}{dt} - i\frac{d\omega_0}{dt}A = 0 \tag{2.20}$$

Adiabatic variation of parameters

We expect that the amplitude A is a slow function of time, and neglect d^2A/dt^2 in this equation, which gives

$$2\omega_0 \frac{dA}{dt} + \frac{d\omega_0}{dt} A = 0 \tag{2.21}$$

This equation can also be written as

$$\frac{d}{dt} \ln(A^2\omega_0) = 0 \tag{2.22}$$

from which it follows that in the adiabatic regime $A(t)^2\omega_0(t) = \mathrm{const}$. We found an *adiabatic invariant* for our oscillator.

Adiabatic variation of parameters

The Figure shows the result of numerical integration of Eq. (2.16) in the adiabatic regime.

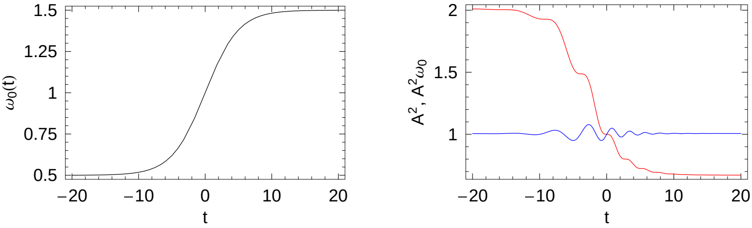


Figure : The left plot shows the function $\omega_0(t)$. The red curve on the right plot shows the quantity $x(t)^2 + \dot{x}^2(t)/\omega_0^2$ (which is close to the amplitude squared A^2) and the blue curve shows the product of this quantity with $\omega_0(t)$. We see that the product is approximately conserved, and hence is an adiabatic invariant.

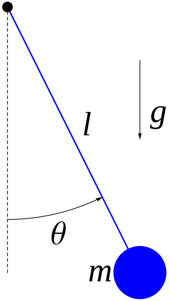
Nonlinear oscillator

The linear oscillator is usually obtained as a first approximation in the expansion near the equilibrium position of a stable system. Higher order terms would lead to nonlinear terms in the equation

$$\frac{d^2x}{dt^2} + \omega_0^2 x = \alpha x^2 + \beta x^3 + \dots \tag{2.23}$$

where the coefficients α, β, are small. What is the effect of these terms? The most important consequence of nonlinear terms is that they introduce a dependance of frequency of oscillations on amplitude.

Pendulum



Instead of studying Eq. (2.23) we will analyze first the *pendulum equation*

$$\ddot{\theta} + \omega_0^2 \sin\theta = 0 \tag{2.24}$$

where $\omega_0^2 = g/l$, l being the length of the pendulum.

Note that for small amplitudes, $\theta \ll 1$, we have

$$\sin\theta \approx \theta - \frac{1}{6}\theta^3 \tag{2.25}$$

and we recover Eq. (2.23) with $\alpha = 0$ and $\beta = \omega_0^2/6$. The linear approximation for the pendulum equation is obtained if we neglect the cubic term in this expansion.

Pendulum

Of course, the pendulum can be solved exactly if we use the energy conservation. Multiplying Eq. (2.24) by $\dot{\theta}$ gives

$$\frac{1}{2}\frac{d}{dt}\dot{\theta}^2 - \omega_0^2 \frac{d}{dt}\cos\theta = 0 \tag{2.26}$$

from which it follows that the quantity

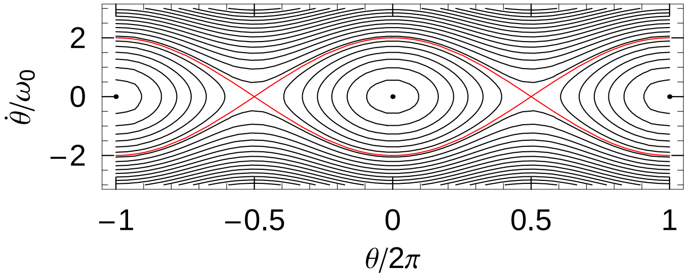
$$E = \frac{1}{2\omega_0^2}\dot{\theta}^2 - \cos\theta = \mathrm{const} \tag{2.27}$$

is conserved. We call E the energy of the system; each orbit is characterized by its own energy. For a given energy E we have

$$\dot{\theta} = \pm\omega_0\sqrt{2(E+\cos\theta)} \tag{2.28}$$

Pendulum

This equation allows us to graph the *phase portrait* of the system where we plot trajectories on the plane $(\theta, \dot{\theta}/\omega_0)$.



There are *stable points*, *unstable points* and the *separatrix* on this plot. Oscillations correspond to values of E such that $-1 < E < 1$, with rotation occuring at $E > 1$. The separatrix has $E = 1$.

Pendulum

Let us find now how the period of the pendulum T (and hence the frequency $\omega = 2\pi/T$) depends on the amplitude. We can integrate Eq. (2.28)

$$\omega_0 \int_{t_1}^{t_2} dt = \int_{\theta_1}^{\theta_2} \frac{d\theta}{\sqrt{2(E + \cos\theta)}} \tag{2.29}$$

For a given energy E, inside the separatrix, the pendulum swings between $-\theta_0$ and θ_0, where θ_0 is defined by the relation $\cos\theta_0 = -E$, hence

$$\omega_0(t_2 - t_1) = \frac{1}{\sqrt{2}} \int_{\theta_1}^{\theta_2} \frac{d\theta}{\sqrt{(\cos\theta - \cos\theta_0)}} \, . \tag{2.30}$$

To find a half a period of the oscillations we need to integrate from $-\theta_0$ to θ_0:

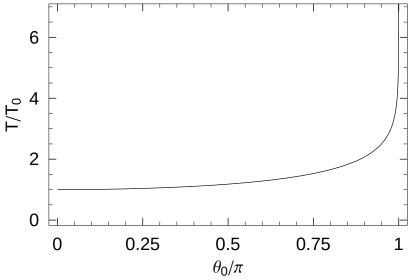
$$\frac{1}{2} T\omega_0 = \frac{1}{\sqrt{2}} \int_{-\theta_0}^{\theta_0} \frac{d\theta}{\sqrt{(\cos\theta - \cos\theta_0)}} \tag{2.31}$$

Pendulum period

The result can be expressed in terms of the elliptic function K of the first kind

$$\frac{T}{T_0} = \frac{2}{\pi} K\left(\sin^2\left(\frac{\theta_0}{2}\right)\right) \qquad (2.32)$$

where $T_0 = 2\pi/\omega_0$ is the period in the linear approximation.
Period T as a function of the amplitude angle θ_0 in the range $0 < \theta_0 < \pi$:



Pendulum period—small amplitudes

For small values of the argument, the Taylor expansion of the elliptic function is: $(2/\pi)K(x) \approx 1 + x/4$. This means that for small amplitudes the frequency of oscillations is given by

$$\omega \approx \omega_0 \left(1 - \frac{\theta_0^2}{16}\right) \qquad (2.33)$$

it decreases with the amplitude.

Pendulum period—large amplitudes

Near the separatrix, the period of oscillations becomes very large. The separatrix corresponds to $\theta_0 = \pi$, and we can use the approximation $(2/\pi)K(1 - x) \approx -(\ln x)/\pi$ valid for $x \ll 1$, to find an approximate formula for T near the separatrix. We obtain

$$\frac{T}{T_0} \approx \frac{1}{\pi} \ln \frac{1}{1 - E} \tag{2.34}$$

As we see, the period diverges logarithmically as E approaches its value at the separatrix.

Nonlinear oscillator

In the general case of Eq. (2.23) the approximate solution will be

$$x(t) = A(t)\cos[\omega(A)t + \phi] \tag{2.35}$$

where the frequency ω now becomes a function of the amplitude

$$\omega(A) \approx \omega_0 + aA^2\,. \tag{2.36}$$

In this equation we have to assume that the correction to the frequency is small, $\omega_0 \gg aA^2$. One can show that

$$a = -\frac{3\beta}{8\omega_0} - \frac{5\alpha^2}{12\omega_0^3} \tag{2.37}$$

Nonlinear resonance

For a linear oscillator a resonant frequency can drive the amplitude to very large values, if the damping is small. The situation changes for a nonlinear oscillator. In this case, when the amplitude grows, the frequency of the oscillator changes and the oscillator detunes itself from the resonance.

Let us first make a rough estimate of the maximum amplitude of a nonlinear resonance. Take Eq. (2.11) and set $\gamma = 0$ (no damping):

$$A^2 = |\xi_0|^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2} \tag{2.38}$$

Replace ω_0 by $\omega_0 + aA^2$ and then set $\omega = \omega_0$ (the frequency of the driving force is equal to that of the *linear* oscillator). Using the smallness of aA^2 we find

$$A^2 \approx \frac{f_0^2}{(2a\omega_0 A^2)^2} \tag{2.39}$$

Nonlinear resonance

This should be considered as an equation for A, from which we find

$$A \approx \left(\frac{f_0}{2a\omega_0} \right)^{1/3} \tag{2.40}$$

We see that due to nonlinearity, even at exact resonance, the amplitude of the oscillations is finite: it is proportional to $a^{-1/3}$.

It is possible for an even larger amplitude A_1 to be reached by gradually increasing ω to $\omega_0 + aA_1^2$.

For consideration

- For an oscillator driven by an external force, how is nonlinearity near resonance similar to damping? How is it different?
- How might you estimate the relative importance of nonlinear dynamics compared to linear damping?
- What are the pros and cons of using an ideal, simple harmonic oscillator as an example of a dynamic system?

Lagrangian and Hamiltonian equations of motion (Lecture 3)

January 25, 2016

Lecture outline

The most general description of motion for a physical system is provided in terms of the Lagrange and the Hamilton functions. In this lecture we introduce the Lagrange equations of motion and discuss the transition from the Lagrange to the Hamilton equations. We write down the Lagrangian and Hamiltonian for a charged particle and introduce the Poisson brackets.
Note: In the 2011 notes, Poisson brackets have the opposite sign convention.

Lagrangian

How to write equations of motion for a complicated mechanical system, like the ones shown below?

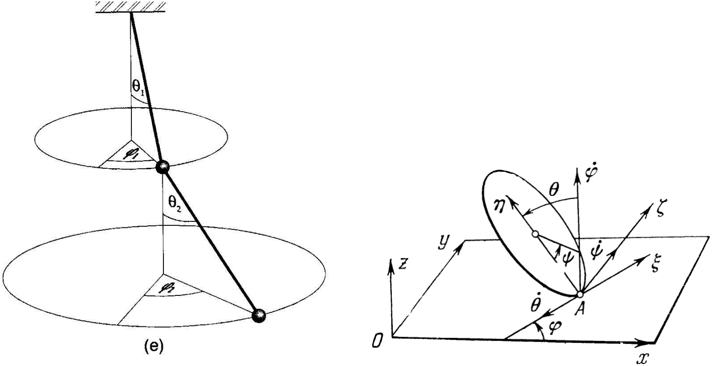


Figure : A spherical pendulum and a rolling disk.

Lagrangian

Each mechanical system is characterized by a Lagrangian function. This function depends on *generalized* coordinates of the system, q_1, q_2, ..., q_n, and velocities $\dot{q}_1$, $\dot{q}_2$,..., $\dot{q}_n$, and time t: $L(q_i, \dot{q}_i, t)$ [for brevity, we will write $L(q_i, \dot{q}_i, t)$ instead of $L(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, t)$]. The number n is the number of *degrees of freedom* for the system.

Lagrangian

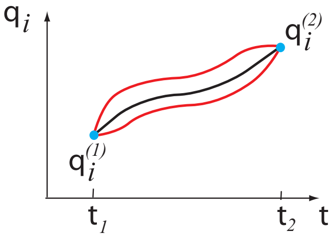


Figure : The action reaches an extremum along the physical trajectory of the system.

The Lagrangian has the following property: the integral

$$S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t) dt \tag{3.1}$$

(which is called the *action*) reaches an extremum along the true trajectory of the system when varies with fixed end points.

This property can be used directly to find trajectories of a system by numerically minimizing the action S. It is however not very practical, in part because the varied trajectory is specified by its initial, $q(t_1)$, and final, $q(t_2)$, positions. In applications we would prefer to specify a trajectory by its initial position and velocity.

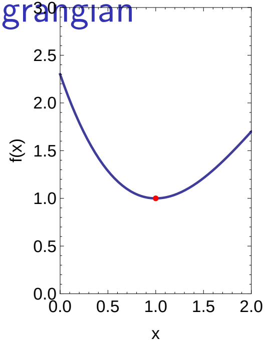
Lagrangian

For mechanical systems, the Lagrangian is equal to the difference between the kinetic energy and the potential energy of the system. For example, for the pendulum, with the angle θ chosen as a generalized coordinate q, the Lagrangian is

$$L(\theta, \dot{\theta}) = \frac{m}{2} l^2 \dot{\theta}^2 + gml \cos\theta \,. \tag{3.2}$$

As was mentioned above, knowing the Lagrangian is enough to be able to find trajectories of the pendulum by direct minimization of the action.

Lagrangian



The most convenient approach to the problem of obtaining equations of motion for a given Lagrangian is based on the variational calculus. By direct minimization of the action integral, requiring

$$\delta \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)\,dt = 0\,, \qquad (3.3)$$

one can get equations of motion in the following form (see proof below):

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0\,, \qquad i = 1, \ldots, n\,. \qquad (3.4)$$

These are ordinary differential equations which are much easier to solve than trying to directly minimize S.

Clarification

What is d/dt? If we have a function $f(q(t), \dot{q}(t), t)$ then

$$\frac{df}{dt} = \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial \dot{q}}\ddot{q} + \frac{\partial f}{\partial t}$$

Given, for example, $g(r(t), t)$

$$\begin{aligned}\frac{dg}{dt} &= \frac{\partial g}{\partial x} \cdot v_x + \frac{\partial g}{\partial y} \cdot v_y + \frac{\partial g}{\partial z} \cdot v_z + \frac{\partial g}{\partial t} \\ &= v \cdot \nabla g + \frac{\partial g}{\partial t}\end{aligned}$$

so we can write in this case

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v \cdot \nabla = \frac{\partial}{\partial t} + v \cdot \frac{\partial}{\partial r}$$

Lagrangian

Let us prove (3.4). I assume 1 degree of freedom. Assume that $q(t)$ is a true orbit and $q(t_1)$ and $q(t_2)$ are fixed. Let $\delta q(t)$ be a deviation from this orbit; it has a property $\delta q(t_1) = \delta q(t_2) = 0$. Compute the variation of the action:

$$
\begin{aligned}
&\delta \int_{t_1}^{t_2} L(q, \dot{q}, t) dt = \\
&= \int_{t_1}^{t_2} L(q + \delta q, \dot{q} + \delta \dot{q}, t) dt - \int_{t_1}^{t_2} L(q, \dot{q}, t) dt \\
&= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) dt = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \delta q dt,
\end{aligned}
\tag{3.5}
$$

where we used $\delta \dot{q} = d\delta q/dt$ and integrated by parts. Since $q(t)$ is a true orbit, the action reaches an extremum on it, and the variation of the action should be of second order, $\propto \delta q^2$. This means that the linear variation that we found above vanishes for arbitrary δq, hence Eq. (3.4) must be satisfied.

Lagrangian

The Lagrangian for a given system is not unique. There exist various forms of the Lagrangian for a physical system that lead to the same equations of motion.
There are several advantages of using Lagrangian as a basic point for formulation of equations of motion: a) easy to choose convenient generalized coordinates, b) it is closely connected to the variational principles, and c) it relates symmetries of the Lagrangian to conservation laws for the system. A disadvantage is that the Lagrangian approach sometimes obscures the nature of the forces acting on the system.
A simple example of the relation between the symmetry of the Lagrangian and the conservation laws is given by the case when L does not depend on q_i. As follows from Eqs. (3.4), in this case the quantity $\partial L/\partial \dot{q}_i$ is conserved.

Relativistic equations of motion in electromagnetic field

For a point charge q moving with velocity v we have

$$\frac{dp}{dt} = qE + qv \times B\,. \tag{3.6}$$

On the right-hand side of this equation we have the *Lorentz force*.

Lagrangian of a relativistic particle in an electromagnetic field

This is the Lagrangian of a relativistic charged particle moving in electromagnetic field represented by the vector potential A and the scalar potential ϕ:

$$L(r, v, t) = -mc^2\sqrt{1 - v^2/c^2} + ev \cdot A(r, t) - e\phi(r, t)\,. \qquad (3.7)$$

In Cartesian coordinate system, $r = (x, y, z)$, and the Lagrangian is given as a function $L(x, y, z, \dot{x}, \dot{y}, \dot{z}, t)$, where, of course, $\dot{x} = v_x$, $\dot{y} = v_y$, $\dot{z} = v_z$.

The electric and magnetic fields are related to the potentials through

$$E = -\nabla\phi - \frac{\partial A}{\partial t}, \qquad B = \nabla \times A \qquad (3.8)$$

Motion in uniform magnetic field

As an example of using the Lagrangian formalism, let us study particle's motion in a uniform magnetic field using the above Lagrangian.
The field is directed along the z-axis:

$$B = (0, 0, B_0)\,. \tag{3.9}$$

It is easy to check that the vector potential can be chosen as

$$A = (-B_0 y, 0, 0)\,, \tag{3.10}$$

so that $B = \nabla \times A$. This gives for the Lagrangian

$$L = -mc^2\sqrt{1 - v^2/c^2} - eB_0 v_x y\,. \tag{3.11}$$

Motion in uniform magnetic field

For the equation of motion in the x direction

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial v_x} = 0$$

we observe that $\partial L/\partial x = 0$ and

$$\frac{\partial L}{\partial v_x} = mc^2\gamma\frac{v_x}{c^2} - eB_0 y$$

which gives

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial v_x} = m\gamma\dot{v}_x - eB_0 v_y = 0\,. \tag{3.12}$$

We introduced $\gamma = (1-\beta^2)^{-1/2}$ with $\beta = v/c$.

Motion in uniform magnetic field

In the last equation we used the fact that the motion in the magnetic field conserves the kinetic energy, $\gamma = \mathrm{const}$, hence $\dot{\gamma} = 0$. The equation for v_x reads

$$\dot{v}_x = \omega_H v_y \,, \tag{3.13}$$

where the *cyclotron* frequency ω_H is

$$\omega_H = \frac{eB_0}{\gamma m} \,. \tag{3.14}$$

Repeating the derivation for the y component gives

$$\dot{v}_y = -\omega_H v_x \,. \tag{3.15}$$

Combining Eqs. (3.13) and (3.15) yields $\ddot{v}_x + {\omega_H}^2 v_x = 0$, with the solution $v_x = v_0 \cos(\omega_H t + \phi_0)$.

Motion in uniform magnetic field

From Eq. (3.13) we then obtain $v_y = -v_0 \sin(\omega_H t + \phi_0)$.
Integrating velocities, we find coordinates:

$$x = \frac{v_0}{\omega_H}\sin(\omega_H t + \phi_0) + x_0\,, \qquad y = \frac{v_0}{\omega_H}\cos(\omega_H t + \phi_0) + y_0\,. \tag{3.16}$$

This is a circular orbit with the radius (*Larmor radius*)

$$R = \frac{v_0}{\omega_H} = \frac{p}{eB_0}\,. \tag{3.17}$$

For the z-direction we have

$$\frac{d}{dt}\frac{\partial L}{\partial v_z} = 0 \qquad \Longrightarrow \qquad \dot{v}_z = 0 \qquad \Longrightarrow \qquad v_z = \mathrm{const}\,. \tag{3.18}$$

From Lagrangian to Hamiltonian

Another way to describe motion of a system is to use the Hamiltonian approach. It has many advantages over the Lagrangian one.
A transition from the Lagrangian to the Hamiltonian is made in three steps. First, we define the *generalized momenta* p_i:

$$p_i = \frac{\partial L(q_k, \dot{q}_k, t)}{\partial \dot{q}_i}\,, \qquad i = 1, \ldots, n\,. \tag{3.19}$$

Second, we solve these equations and express $\dot{q}_i$ in terms of q_i, p_i and t

$$\dot{q}_i = \dot{q}_i(p_k, q_k, t)\,, \qquad i = 1, \ldots, n\,. \tag{3.20}$$

From Lagrangian to Hamiltonian

Third, we make a *Hamiltonian function H*,

$$H = \left(\sum_{i=1}^{n} p_i \dot{q}_i \right) - L(q_k, \dot{q}_k, t)\,, \tag{3.21}$$

and express all $\dot{q}_i$ on the right hand side through q_i, p_i and t using Eqs. (3.20) so that we get the Hamiltonian as a function of variables q_i, p_i and t: $H(q_1, p_1, \ldots, q_n, p_n, t)$.
With the Hamiltonian, the equations of motion become:

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}\,, \qquad \dot{q}_i = \frac{\partial H}{\partial p_i}\,. \tag{3.22}$$

The variables p_i and q_i are called the *canonically conjugate* variables.

From Lagrangian to Hamiltonian

Let us prove (3.22). I assume one degree of freedom. Keep in mind that $L(q, \dot{q}(q, p, t), t)$

$$
\begin{aligned}
-\left(\frac{\partial H}{\partial q}\right)_p &= -\frac{\partial}{\partial q}\left(p\dot{q} - L\right) \\
&= \left(-p\frac{\partial \dot{q}}{\partial q} + \frac{\partial L}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial q}\right) + \frac{\partial L}{\partial q} = \frac{\partial L}{\partial q} \\
&= \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{dp}{dt} \qquad (3.23)
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{\partial H}{\partial p}\right)_q &= \frac{\partial}{\partial p}\left(p\dot{q} - L\right) \\
&= \dot{q} + \left(p\frac{\partial \dot{q}}{\partial p} - \frac{\partial L}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial p}\right) = \dot{q} \qquad (3.24)
\end{aligned}
$$

Hamiltonian of a charged particle in an electromagnetic field

We start from the Lagrangian (3.7)

$$L(r, v, t) = -mc^2\sqrt{1 - v^2/c^2} + ev \cdot A(r, t) - e\phi(r, t),$$

where $r = (x, y, z)$ and $v = (v_x, v_y, v_z)$. First, we need to find the canonical conjugate momentum which we denote by $\boldsymbol{\pi}$ combining into vector notation three cartesian coordinates (π_x, π_y, π_z):

$$\begin{aligned}
\boldsymbol{\pi} &= \frac{\partial L}{\partial v} \\
&= -mc^2 \frac{\partial \sqrt{1 - v^2/c^2}}{\partial v} + eA \\
&= m\frac{v}{\sqrt{1 - v^2/c^2}} + eA = m\gamma v + eA\,. \qquad (3.25)
\end{aligned}$$

Note that the conjugate momentum $\boldsymbol{\pi}$ differs from the kinetic particle's momentum $m\gamma v$.

Hamiltonian of a charged particle in an electromagnetic field

Note that as follows from the previous equation, $\gamma\boldsymbol{\beta} = (\boldsymbol{\pi} - eA)/mc$, and hence

$$\gamma^2\beta^2 = \frac{(\boldsymbol{\pi} - eA)^2}{m^2c^2} \,. \tag{3.26}$$

Now let us derive the Hamiltonian

$$\begin{aligned} H &= v \cdot \boldsymbol{\pi} - L \\ &= v \cdot \boldsymbol{\pi} + mc^2\sqrt{1 - v^2/c^2} - ev \cdot A + e\phi \\ &= m\gamma v^2 + \frac{mc^2}{\gamma} + e\phi \\ &= m\gamma c^2 \left(\beta^2 + \frac{1}{\gamma^2}\right) + e\phi \\ &= m\gamma c^2 + e\phi \,. \end{aligned} \tag{3.27}$$

Hamiltonian of a charged particle in an electromagnetic field

The Hamiltonian is the sum of the energy $m\gamma c^2$ and the potential energy associated with the electrostatic potential ϕ. The vector potential is hidden in Eq. (3.27). To see that, remember, the we need to express H in terms of the conjugate coordinates and momenta. Using Eq. (3.26) we obtain:

$$\gamma^2 = 1 + \gamma^2\beta^2 = 1 + \frac{(\boldsymbol{\pi} - eA)^2}{m^2c^2}\,, \tag{3.28}$$

which gives for the Hamiltonian

$$H = \sqrt{(mc^2)^2 + c^2(\boldsymbol{\pi} - eA)^2} + e\phi\,. \tag{3.29}$$

This is the Hamiltonian of a charged particle in electromagnetic field.

Poisson brackets

Let $f(q_i, p_i, t)$ be a function of coordinate, momenta and time. Assume that coordinates and momenta evolve according to the Hamilton equations, and $q_i(t)$ and $p_i(t)$ represent a trajectory. Then f becomes a function of time t only: $f(q_i(t), p_i(t), t)$. What is the derivative of this function with respect to time? We have

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_i \left(\frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial p_i} \dot{p}_i \right) . \tag{3.30}$$

Substituting Eqs. (3.22) into these equations gives

$$\begin{aligned} \frac{df}{dt} &= \frac{\partial f}{\partial t} + \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial H}{\partial q_i} \right) \\ &= \frac{\partial f}{\partial t} + \{f, H\}, \end{aligned} \tag{3.31}$$

where we introduced the *Poisson brackets*

$$\{f, H\} = \sum_i \left(\frac{\partial H}{\partial p_i} \frac{\partial f}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial f}{\partial p_i} \right) . \tag{3.32}$$

Poisson brackets

Poisson brackets have many remarkable properties. We will use the following two in the next lectures. For two functions $f(q_i, p_i, t)$ and $g(q_i, p_i, t)$

$$\{g, f\} = -\{f, g\}, \tag{3.33}$$

and also

$$\{f, f\} = 0. \tag{3.34}$$

It is easy to verify that following identities hold

$$\{q_i, q_k\} = \{p_i, p_k\} = 0, \qquad \{q_i, p_k\} = \delta_{ik}. \tag{3.35}$$

If the Hamiltonian $H(q_i(t), p_i(t), t)$ is considered as a function of time then

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \{H, H\} = \frac{\partial H}{\partial t} \tag{3.36}$$

Canonical transformations (Lecture 4)

January 26, 2016

Lecture outline

We will introduce and discuss canonical transformations that conserve the Hamiltonian structure of equations of motion. Poisson brackets are used to verify that a given transformation is canonical. A practical way to devise canonical transformation is based on usage of generation functions.
The motivation behind this study is to understand the freedom which we have in the choice of various sets of coordinates and momenta. Later we will use this freedom to select a convenient set of coordinates for description of partilcle's motion in an accelerator.

Introduction

Within the Lagrangian approach we can choose the generalized coordinates as we please. We can start with a set of coordinates q_i and then introduce generalized momenta p_i according to Eqs.

$$p_i = \frac{\partial L(q_k, \dot{q}_k, t)}{\partial \dot{q}_i}, \qquad i = 1, \ldots, n,$$

and form the Hamiltonian

$$H = \left(\sum_i p_i \dot{q}_i \right) - L(q_k, \dot{q}_k, t).$$

Or, we can chose another set of generalized coordinates $Q_i = Q_i(q_k, t)$, express the Lagrangian as a function of Q_i, and obtain a different set of momenta P_i and a different Hamiltonian $H'(Q_i, P_i, t)$. This type of transformation is called a *point transformation*. The two representations are physically equivalent and they describe the same dynamics of our physical system.

Introduction

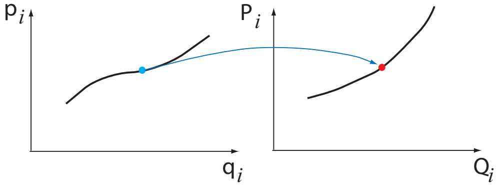
A more general approach to the problem of using various variables in Hamiltonian formulation of equations of motion is the following. Let us assume that we have canonical variables q_i, p_i and the corresponding Hamiltonian $H(q_i, p_i, t)$ and then make a transformation to new variables

$$Q_i = Q_i(q_k, p_k, t)\,, \qquad P_i = P_i(q_k, p_k, t)\,. \qquad i = 1 \dots n. \quad (4.1)$$

Can we find a new Hamiltonian $H'(Q_i, P_i, t)$ such that the system motion in new variables satisfies Hamiltonian equations with H'? What are requirements on the transformation (4.1) for such a Hamiltonian to exist?
These questions lead us to *canonical transformations*. They are more general than point transformations and they can depend on $\dot{q}$.

Canonical transformations

We first consider a time independent Hamiltonian H, and later generalize the result for the case when H is a function of time. Let us assume that we have canonical variables q_i, p_i and the Hamiltonian $H(q_i, p_i)$.



Instead of q_i, p_i we would like to use a new set of independent variables Q_i, P_i that are related to the old one,

$$Q_i = Q_i(q_k, p_k)\,, \qquad P_i = P_i(q_k, p_k)\,, \qquad i = 1 \ldots n. \tag{4.2}$$

Canonical transformations

We assume that the inverse transformation from Q_i, P_i to q_i, p_i exists and write it as follows

$$q_i = q_i(Q_k, P_k)\,, \qquad p_i = p_i(Q_k, P_k)\,, \qquad i = 1 \ldots n. \tag{4.3}$$

It is obtained by considering Eqs. (4.2) as $2n$ equations for the old variables and solving them for q_i, p_i.
For simplicity I consider $n = 1$ and drop indexes. Let us introduce a function H' which is H expressed in terms of new variables (we do not know yet if H' can be used as a Hamiltonian) :

$$H'(Q, P) = H(q(Q, P), p(Q, P))\,. \tag{4.4}$$

Let us assume that we solved Hamiltonian equations of motion and found a trajectory $q(t)$, $p(t)$. This trajectory gives us an orbit in new variables as well, through the transformation (4.2):

$$Q(t) = Q(q(t), p(t))\,, \qquad P(t) = P(q(t), p(t))\,. \tag{4.5}$$

Canonical transformations

We would like the trajectory defined by the functions $Q(t)$ and $P(t)$ to be a Hamiltonian orbit, that is to say that we would like it to satisfy the equations

$$\frac{dP}{dt} = -\frac{\partial H'(Q(t), P(t))}{\partial Q}\,, \qquad \frac{dQ}{dt} = \frac{\partial H'(Q(t), P(t))}{\partial P}\,. \tag{4.6}$$

If these conditions are satisfied for *every Hamiltonian H*, then Eqs. (4.2) give us a *canonical transformation*.
Here are two trivial examples of canonical transformations:

$$Q = p\,, \qquad P = -q\,. \tag{4.7}$$

$$Q = -p\,, \qquad P = q\,. \tag{4.8}$$

When is a transformation canonical?

How to find out if a given transformation (4.2) is canonical? It turns out that it is canonical if and only if

$$\{Q, Q\}_{q,p} = 0\,, \tag{4.9}$$

$$\{P, P\}_{q,p} = 0\,, \tag{4.10}$$

$$\{Q, P\}_{q,p} = 1\,. \tag{4.11}$$

Proof is in the lecture notes. Note that the form of the Hamiltonian is not used, except to define the original canonical momenta.

Example: Point transformation

Given $Q = f_Q(q, t)$, then new momentum $P = p/(\partial f_Q/\partial q)$.
Note that if $p = 0$, then $P = 0$. We can apply a shift in coordinate system, but momentum still has a sense of proportionality. In particular, an object at rest should have zero velocity in any coordinate system.
We have

$$\{P, Q\}_{q,p} = \frac{\partial P}{\partial p}\frac{\partial Q}{\partial q} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} = \frac{1}{\partial f_Q/\partial q}\frac{\partial f_Q}{\partial q} + 0 = 1 \ .$$

This is a small subset of possible canonical transformations.

Example: Linear combination

Given:

$$Q = aq + bp, \qquad P = cq + dp.$$

When this variable transformation is canonical? Compute Poisson brackets

$$\{Q, Q\}_{q,p} = \{P, P\}_{q,p} = 0$$

$$\{Q, P\}_{q,p} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = ad - cb = 1$$

The determinant of the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

should be equal to 1.

Generating functions

Poisson brackets do not give us a method to generate canonical transformations. The technique which allows one to create a transformation that is guaranteed to be canonical is based on the so called *generating functions*.

We will give a complete formulation of the method of generation functions in the next section. Here, we consider a special case of a generating function which is a function of coordinates q_i and new coordinated Q_i. Let us denote such a function by F_1,

$$F_1(q_i, Q_i)\,. \tag{4.12}$$

The canonical transformation associated with this function is given by the following equations:

$$p_i = \frac{\partial F_1(q_i, Q_i)}{\partial q_i}\,, \qquad P_i = -\frac{\partial F_1(q_i, Q_i)}{\partial Q_i}\,, \qquad i = 1 \dots n\,. \tag{4.13}$$

Generating functions

These equations should be considered as equations for Q_i and P_i. Solving the equations we find the transformation (4.2). It turns out that this transformation is canonical.

We will prove that Eqs. (4.13) define a canonical transformation in the case of one degree of freedom and time-independent. In this case, we have only two conjugate variables, q and p, and a canonical transformation (4.2) is given by two equations

$$Q = Q(q,p)\,, \qquad P = P(q,p)\,. \tag{4.14}$$

The generating function $F_1(q,Q)$ is a function of two variables, and from (4.13) we have

$$p = \frac{\partial F_1(q,Q)}{\partial q}\,, \qquad P = -\frac{\partial F_1(q,Q)}{\partial Q}\,. \tag{4.15}$$

Proof for 1 degree of freedom

We first note that for one degree of freedom, the conditions $\{Q, Q\} = 0$ and $\{P, P\} = 0$ are trivially true, because there are no cross terms from other degrees of freedom. We only need to prove

$$\{Q, P\}_{q,p} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = 1\,. \tag{4.16}$$

We now have to remember to characterize the generating function as $F_1(q, Q(q, p))$. From the second of Eqs. (4.15) we have

$$\frac{\partial P}{\partial q} = -\frac{\partial^2 F_1}{\partial Q \partial q} - \frac{\partial^2 F_1}{\partial Q^2}\frac{\partial Q}{\partial q}\,, \qquad \frac{\partial P}{\partial p} = -\frac{\partial^2 F_1}{\partial Q^2}\frac{\partial Q}{\partial p}\,. \tag{4.17}$$

Proof for 1 degree of freedom (cont'd)

Substituting these equation into Eq. (4.16) we obtain

$$\frac{\partial P}{\partial p}\frac{\partial Q}{\partial q} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} = \frac{\partial^2 F_1}{\partial Q \partial q}\frac{\partial Q}{\partial p}\,. \tag{4.18}$$

The derivative $\partial Q/\partial p$ can be found when we differentiate the first of equations (4.15) with respect to p:

$$1 = \frac{\partial^2 F_1}{\partial q \partial Q}\frac{\partial Q}{\partial p}\,, \tag{4.19}$$

and we see that RHS of (4.18) is equal to 1, which means

$$\{Q, P\}_{q,p} = 1\,. \tag{4.20}$$

Four types of generating functions

The functions (4.12) are not the only type of functions that generate canonical transformations. Below we will list other types that can be used for this purpose. Before that, however, we need to generalize our result for the time dependent transformations and time dependent Hamiltonians.

Four types of generating functions

Canonical transformations can be time dependent,

$$Q_i = Q_i(q_k, p_k, t)\,, \qquad P_i = P_i(q_k, p_k, t)\,. \qquad i = 1 \ldots n. \quad (4.21)$$

They can be applied to time dependent Hamiltonians as well. The Poisson brackets are still applicable in this case, and Eqs. (4.9) - (4.11) are necessary and sufficient conditions for a transformation to be canonical (the variable t can be considered as a parameter in calculation of the Poisson brackets). However, the simple rule (4.4) for obtaining a new Hamiltonian needs to be modified for the general case of time dependent transformations.

Four types of generating functions

Previously we introduced the generating function $F_1(q_i, Q_i)$. This is only one of four possible types of the generating functions. All generating functions depend on a set of old coordinates and a set of new ones. These sets come in the following combinations: (q_i, Q_i), (q_i, P_i), (p_i, Q_i), and (p_i, P_i). Correspondingly, we have 4 types of the generating function. The rules how to make a canonical transformation for each type of the generating function, and the associated transformation of the Hamiltonian, are shown below.

Four types of generating functions

The first type of the generating functions is $F_1(q_i, Q_i, t)$:

$$p_i = \frac{\partial F_1}{\partial q_i}\,, \qquad P_i = -\frac{\partial F_1}{\partial Q_i}\,,$$
$$H' = H + \frac{\partial F_1}{\partial t}\,. \tag{4.22}$$

The second type is $F_2(q_i, P_i, t)$:

$$p_i = \frac{\partial F_2}{\partial q_i}\,, \qquad Q_i = \frac{\partial F_2}{\partial P_i}\,,$$
$$H' = H + \frac{\partial F_2}{\partial t}\,. \tag{4.23}$$

Four types of generating functions

The third type is $F_3(p_i, Q_i, t)$:

$$q_i = -\frac{\partial F_3}{\partial p_i}\,, \qquad P_i = -\frac{\partial F_3}{\partial Q_i}\,,$$
$$H' = H + \frac{\partial F_3}{\partial t}\,. \tag{4.24}$$

The fourth type is $F_4(p_i, P_i, t)$:

$$q_i = -\frac{\partial F_4}{\partial p_i}\,, \qquad Q_i = \frac{\partial F_4}{\partial P_i}\,,$$
$$H' = H + \frac{\partial F_4}{\partial t}\,. \tag{4.25}$$

Example of canonical transformations

We first consider a simple example of the identity transformation

$$Q_i = q_i \,, \qquad P_i = p_i \,. \tag{4.26}$$

The generating function of the second type for this transformation is

$$F_2 = \sum_{i=1}^{n} q_i P_i \,. \tag{4.27}$$

Revisit point transformations

We now go back to point transformations, $Q_j = f_{Qj}(\{q_k\}, t)$. We will show that

$$F_2 = \sum_{i=1}^{n} f_{Qi}(\{q_k\}, t) P_i \tag{4.28}$$

and that the dynamics for the modified Langrangian agrees with the additional term $\partial F_2/\partial t$.
The generating function yields

$$p_j = \frac{\partial F_2}{\partial q_j} = \sum_{i=1}^{n} \frac{\partial f_{Qi}}{\partial q_j} P_i = \sum_{i=1}^{n} W_{ij} P_i \ , \tag{4.29}$$

where $W_{ij} \equiv \partial f_{Qi}/\partial q_j$. Therefore,

$$P_j = \sum_{i=1}^{n} p_i W_{ij}^{-1} \tag{4.30}$$

Revisit point transformations

The new Hamiltonian is given by

$$H' = H + \frac{\partial F_2}{\partial t} = H + \sum_{i=1}^{n} \frac{\partial f_{Qi}}{\partial t} P_i \ . \tag{4.31}$$

In terms of the original Lagrangian, we have the same action integral (it gets a little more complicated for more general cases, add a total time derivative to L'):

$$S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)\ dt = \int_{t_1}^{t_2} L'(Q_i, \dot{Q}_i, t)\ dt \tag{4.32}$$

so $L = L'$. Also, by definition of the transformation,

$$\dot{Q}_i = \sum_{k=1}^{n} \frac{\partial f_{Qi}}{\partial q_k} \dot{q}_k + \frac{\partial f_{Qi}}{\partial t} \ . \tag{4.33}$$

Revisit point transformations

If the original conjugate momentum was $p_i = \partial L/\partial \dot{q}_i$, then the new conjugate momentum has to satisfy

$$p_j = \frac{\partial L}{\partial \dot{q}_j} = \sum_{i=1}^{n} \frac{\partial L'}{\partial \dot{Q}_i}\frac{\partial \dot{Q}_i}{\partial \dot{q}_j} = \sum_{i=1}^{n} \frac{\partial L'}{\partial \dot{Q}_i}\frac{\partial Q_i}{\partial q_j} = \sum_{i=1}^{n} P_i W_{ij} \ . \qquad (4.34)$$

This is exactly what we found from the generating function. Also, the new Hamiltonian is

$$\begin{aligned} H' = \sum_{i=1}^{n} P_i \dot{Q}_i - L' &= \sum_{i=1}^{n}\sum_{j=1}^{n} p_j W_{ji}^{-1}\left(\sum_{k=1}^{n} \dot{q}_k W_{ik} + \frac{\partial f_{Qi}}{\partial t}\right) - L \\ &= \sum_{j=1}^{n} p_j \dot{q}_j + \sum_{i=1}^{n}\sum_{j=1}^{n} p_j W_{ji}^{-1}\frac{\partial f_{Qi}}{\partial t} - L = H + \frac{\partial F_2}{\partial t} \ . \qquad (4.35)\end{aligned}$$

So at least for point transformations, the formulas for generating functions match the requirements for going from a Lagrangian to a Hamiltonian exactly.

Simple point transformations: swap x and y; rotations

A simple point transformation is to swap two physical coordinates. To keep the notation similar, we use $x = q_1$ and $y = q_2$, and include q_3 as an extra degree of freedom. From Eq. (4.28) we see that the generating function is

$$F_2 = q_2 P_1 + q_1 P_2 + q_3 P_3 \tag{4.36}$$

A rotation in the $x - y$ plane by θ works in a similar way,

$$F_2 = [(\cos\theta)q_1 + (\sin\theta)q_2]P_1 + [(\cos\theta)q_2 - (\sin\theta)q_1]P_2 + q_3 P_3 \tag{4.37}$$

Note that $\theta = \pi/2$ gives a slightly different swap because y maps to $-x$. Furthermore, if θ has an explicit time-dependence, the new Hamiltonian will have an additional term which reduces to

$$H' = H + \frac{\partial F_2}{\partial t} = H + \dot{\theta}(Q_2 P_1 - Q_1 P_2) \tag{4.38}$$

Action-angle variables for linear oscillator

We now show how canonical transformations can be applied to the harmonic oscillator. The Hamiltonian for an oscillator with a unit mass is

$$H = \frac{p^2}{2} + \frac{\omega^2 x^2}{2}\,. \tag{4.39}$$

It gives the following equations of motion: $\dot{p} = -\partial H/\partial x = \omega^2 x$, $\dot{x} = \partial H/\partial p = p$ with the solution

$$x = a\cos(\omega t + \phi_0)\,, \qquad p = -a\omega \sin(\omega t + \phi_0)\,. \tag{4.40}$$

We would like to introduce a set of new variables, J (new momentum) and ϕ (new coordinate), such that the transformation from new to old coordinates would be

$$x = a(J)\cos\phi\,, \qquad p = -a(J)\omega\sin\phi\,. \tag{4.41}$$

The advantage of the new variables is clear: the new momentum J is a constant of motion (because $a = \mathrm{const}$), and the new coordinate evolves in a simple way, $\phi = \omega t + \phi_0$.

Action-angle variables for linear oscillator

We will use a generating function $F_1(x, \phi)$ of the first type. To find this function, we need to express p in terms of the old and new coordinates

$$p = -\omega x \tan\phi \,. \tag{4.42}$$

Integrating the equation $p = (\partial F_1/\partial x)_\phi$, we find

$$F_1(x, \phi) = \int^x p d\tilde{x} = -\frac{\omega x^2}{2} \tan\phi \,. \tag{4.43}$$

Aside on generating functions

In general, the generating function needs to satisfy

$$F = \int^{q} p \; d\tilde{q} \qquad \text{for } F_1 \text{ or } F_2 \tag{4.44}$$

or

$$F = -\int^{p} q \; d\tilde{p} \qquad \text{for } F_3 \text{ or } F_4 \tag{4.45}$$

For F_1, Q will be held fixed. For F_2, P will be held fixed, which is more "natural" for going to action-angle variables.
The math is less tedious using F_1, that is the only motivation for using this form. In fact there is a relation between the two; given any F_1, we can get the same result using $F_2(q, P) = \sum PQ + F_1$. This is a type of Legendre transform.

Action-angle variables for linear oscillator

We have

$$J = -\frac{\partial F_1}{\partial \phi} = \frac{\omega x^2}{2}\frac{1}{\cos^2\phi} = \frac{\omega x^2}{2}(1+\tan^2\phi) = \frac{\omega x^2}{2}\left(1+\frac{p^2}{\omega^2 x^2}\right) = \frac{1}{2\omega}\left(\omega^2 x^2 + p^2\right) . \tag{4.46}$$

This equation expresses the new momentum in terms of the old variables. The new coordinate can be found from Eq. (4.42)

$$\phi = -\arctan\frac{p}{\omega x} . \tag{4.47}$$

Action-angle variables for linear oscillator

The new Hamiltonian is a function of the new momentum only

$$H = \omega J\,, \tag{4.48}$$

and gives the following equations of motion in new variables:

$$\dot{J} = -\frac{\partial H}{\partial \phi} = 0\,, \qquad \dot{\phi} = \frac{\partial H}{\partial J} = \omega\,. \tag{4.49}$$

The oscillator dynamics looks very simple in new coordinates:

$$J = \mathrm{const}\,, \qquad \phi = \omega t + \phi_0\,. \tag{4.50}$$

The (J, ϕ) pair is called the *action-angle* coordinates for this particular case. They are very useful for building a perturbation theory in a system which in the zeroth approximation reduces to a linear oscillator.

Action-angle variables for linear oscillator

Here are expressions for old variables in terms of the new ones

$$
\begin{aligned}
x &= \sqrt{\frac{2J}{\omega_0}}\cos\phi \\
p &= -\sqrt{2J\omega_0}\sin\phi
\end{aligned}
\tag{4.51}
$$

This follows from (4.46) and (4.47).

Hamiltonian flow in phase space and Liouville's theorem (Lecture 5)

January 26, 2016

Lecture outline

We will discuss the Hamiltonian flow in the phase space. This flow represents a time dependent canonical transformation of the Hamiltonian variables.
We then discuss conservation of the phase volume in the Hamiltonian flow (Liouville's theorem).

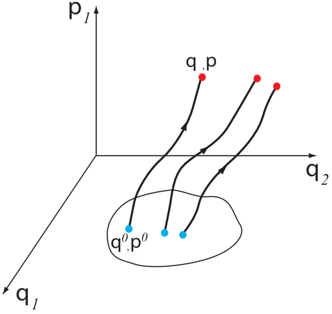
Hamiltonian flow in phase space

We will now take another look at the Hamiltonian motion focusing on its geometrical aspect. Let us assume that for a Hamiltonian $H(q_i, p_i, t)$, for every set of initial condition p_i^0, q_i^0 from some domain at time t_0 we can solve the equations of motion starting from time t_0 and find the values p_i, q_i at time t. This gives us a *map*

$$p_i = p_i(p_i^0, q_i^0, t_0, t)\,, \qquad q_i = q_i(p_i^0, q_i^0, t_0, t)\,. \tag{5.1}$$

Hamiltonian flow in phase space

Considering the time t as a variable we will move each point (q_i, p_i) along a trajectory in the phase space. A collection of such trajectories with various initial conditions (q_i^0, p_i^0) constitutes a *Hamiltonian flow*.



Hamiltonian flow is canonical

A remarkable feature of the relations (5.1) is that, for a given t_0 and t, they constitute a canonical transformation from p_i^0, q_i^0 to p_i, q_i, which is also called a *symplectic* transfer map. We will prove the canonical properties of this map for one degree of freedom. In this case we can drop the index $p_i \longrightarrow p$, $q_i \longrightarrow q$, etc. We will also focus only on one of the Poisson brackets, $\{q, p\}_{q_0,p_0}$; the two others can be analyzed in a similar way.

One degree of freedom

In our proof we demonstrate that the time derivative of the Poisson bracket $\{p, q\}_{q_0,p_0}$ is equal to zero at $t = t_0$. Since t_0 can be considered as an arbitrary moment of time, this means that the time derivative of the Poisson bracket is identically equal to zero at all times. And the conservation of the Poisson brackets in time means that the map (5.1) remains symplectic for all values of t. Note that at $t = t_0$ the map (5.1) becomes an identity transformation, $p = p_0$, $q = q_0$, and obviously

$$\{q, p\}_{q_0,p_0}\Big|_{t=t_0} = 1\,. \tag{5.2}$$

Proof

We only calculate the time derivative of $\{q, p\}_{q_0,p_0}$; the other brackets can be analyzed in a similar way. We have

$$\frac{d}{dt}\{q,p\}_{q_0,p_0} = \frac{d}{dt}\left(\frac{\partial q}{\partial q_0}\frac{\partial p}{\partial p_0} - \frac{\partial q}{\partial p_0}\frac{\partial p}{\partial q_0}\right) \tag{5.3}$$

$$= \frac{\partial q}{\partial q_0}\frac{\partial}{\partial p_0}\frac{dp}{dt} + \frac{\partial p}{\partial p_0}\frac{\partial}{\partial q_0}\frac{dq}{dt} - \frac{\partial q}{\partial p_0}\frac{\partial}{\partial q_0}\frac{dp}{dt} - \frac{\partial p}{\partial q_0}\frac{\partial}{\partial p_0}\frac{dq}{dt}$$

$$= -\frac{\partial q}{\partial q_0}\frac{\partial}{\partial p_0}\frac{\partial H}{\partial q} + \frac{\partial p}{\partial p_0}\frac{\partial}{\partial q_0}\frac{\partial H}{\partial p} + \frac{\partial q}{\partial p_0}\frac{\partial}{\partial q_0}\frac{\partial H}{\partial q} - \frac{\partial p}{\partial q_0}\frac{\partial}{\partial p_0}\frac{\partial H}{\partial p}\,.$$

Using the chain rules for calculation of the partial derivatives

$$\frac{\partial}{\partial p_0} = \frac{\partial p}{\partial p_0}\frac{\partial}{\partial p} - \frac{\partial q}{\partial p_0}\frac{\partial}{\partial q}, \qquad \frac{\partial}{\partial q_0} = \frac{\partial p}{\partial q_0}\frac{\partial}{\partial p} - \frac{\partial q}{\partial q_0}\frac{\partial}{\partial q} \tag{5.4}$$

it is tedious but straightforward to show that all the terms on the right hand side of (5.3) cancel out and $d\{q,p\}_{q_0,p_0}/dt = 0$.

Symplectic maps

In accelerator context we might have a beam diagnostic at one location of the ring, which measures coordinates of a bunched beam when it passes by at time t_0. On the next turn, at time $t = t_0 + T$, where T is the revolution period in the ring, it measures coordinates again. The relation between the new and the old coordinates for various particles in the beam will be given by the functions (5.1).

Symplectic maps

The different language of *symplectic maps* is often used in connection with this type of canonical transformation. A linear map M is defined as symplectic if

$$MJ_{2n}M^T = J_{2n}\,, \tag{5.5}$$

with

$$J_{2n} = \begin{pmatrix} J_2 & 0 & 0 & 0 \\ 0 & J_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & J_2 \end{pmatrix}, \qquad \text{and} \qquad J_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{5.6}$$

It is clear the determinant of M must be equal to ± 1. What is more difficult to see is that only $\det M = 1$ is possible.
This is not a sufficient condition except for special case of 1 degree of freedom.

Symplectic maps

Let us consider the transformation $Q_i = Q_i(q_k, p_k)$, $P_i = P_i(q_k, p_k)$, $i = 1 \dots n$, and change the notation introducing $w_{2k-1} = q_k$, $w_{2k} = p_k$, $W_{2k-1} = Q_k$, $W_{2k} = P_k$, $k = 1, 2, \dots n$.
For examples, for $n = 2$ we have $w_1 = q_1$, $w_2 = p_1$, $w_3 = q_2$, $w_4 = p_2$, and the same set of relations with small letters replaced by the capital ones.
A transformation from old to new variables is then given by $2n$ functions

$$W_i = W_i(w_k), \qquad i, k = 1, 2, \dots 2n\,. \tag{5.7}$$

Symplectic maps

It turns out that the requirement that all possible Poisson brackets satisfy $\{Q_i, Q_k\}_{q,p} = 0$, $\{P_i, P_k\}_{q,p} = 0$, $\{Q_i, P_k\}_{q,p} = \delta_{ik}$, (which is equivalent to the requirement for the transformation to be canonical) can be written as

$$MJ_{2n}M^T = J_{2n}\,, \tag{5.8}$$

where M is the Jacobian matrix of the transformation (with the elements $M_{i,j} = \partial W_i/\partial w_j$) and the superscript T denotes transposition of a matrix. This must hold true throughout phase space.
The transformation can be nonlinear, but locally about *any* region in phase space it must look like a symplectic linear map.

Liouville's theorem

A general Hamiltonian flow in the phase space conserves several integrals of motion. The most important one is the volume occupied by an ensemble of particles. Conservation of the phase space volume is called the *Liouville theorem*.

The phase space volume is expressed as a $2n$-dimensional integral

$$V_1 = \int_{\mathcal{M}_1} dq_1 dq_2 \ldots dq_n dp_1 dp_2 \ldots dp_n \,, \tag{5.9}$$

where the integration goes over a $2n$-dimensional manifold $\mathcal{M}_1$ in the phase space. A canonical transformation maps the manifold onto a different one $\mathcal{M}_2$, and the new volume phase space is

$$V_2 = \int_{\mathcal{M}_2} dQ_1 dQ_2 \ldots dQ_n dP_1 dP_2 \ldots dP_n \,. \tag{5.10}$$

Liouville's theorem

The ratio of elementary volumes, as is known from the mathematical analysis is equal to the determinant of the Jacobian of the transformation M

$$\left|\frac{dQ_1 dQ_2 \ldots dQ_n dP_1 dP_2 \ldots dP_n}{dq_1 dq_2 \ldots dq_n dp_1 dp_2 \ldots dp_n}\right| = |\mathrm{det} M| \,. \tag{5.11}$$

Using Eq. (5.5) it is easy to prove that $|\mathrm{det} M| = 1$.

Extended canonical transformations

It is useful to define an *extended* Hamiltonian by including not just time but the energy itself as variables, arranged in a pair with time acting as the coordinate and energy (almost) as the conjugate momentum.

$$H_{\text{ext}}(q, p, t, p_t) \equiv H(q, p, t) + p_t \tag{5.12}$$

with $p_t = -H(q, p, t)$ required as a constraint. Usually we write $p_t = -h$; be careful with signs.

Now we have the simple constraint $H_{\text{ext}} = 0$. New equations of motion are $dt/dt = 1$, $dh/dt = dH/dt = \partial H/\partial t$. No change.

What we get is a more general class of canonical transformations that can explicitly involve time and energy and alter them.

Generating functions work the same way, with extra minus signs when h is used instead of p_t.

Interpretation of extended phase space

We started with the variational principle to minimize

$$S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)\, dt \tag{5.13}$$

A simple co-ordinate change yields

$$S = \int_{\tau_1}^{\tau_2} L\left[q_i(\tau), \frac{q_i'(\tau)}{t'(\tau)}, t(\tau)\right] t'(\tau)\ d\tau\ , \tag{5.14}$$

where $t' \equiv dt/d\tau$ and $q_i' \equiv dq_i/d\tau = t' dq_i/dt = t'\dot{q}_i$. Now we can treat time t as just another co-ordinate, on an equal footing with the others. This gives us a new Lagrangian with an extra degree of freedom,

$$L_{\text{ext}}(q_i, t, q_i', t') = L\left(q_i, \frac{q_i'}{t'}, t\right) t'\ . \tag{5.15}$$

Example: scale change

The simplest transformation is to redefine time according to $\hat{t} = f_T(t)$. The generating function is $F_2 = f_T(t)\hat{h}$. Then

$$\begin{aligned}
\hat{H}_{\text{ext}}(p, q, \hat{t}, -\hat{h}) &= [H(p, q, t) - h]\,\frac{df_T}{dt} \\
&= \left[H(p, q, t)\frac{df_T}{dt} - \hat{h}\right] \ . \qquad (5.16)
\end{aligned}$$

This is consistent with the conjugate to $\hat{t}$ being $h\,dt/d\hat{t}$, similar to how the Lagrangian changes with a rescaling of time.

Example: swapping a co-ordinate with time

The idea for replacing the independent variable with a co-ordinate is simple. Since time looks like a co-ordinate, it is (up to a sign) the same type of generating function as a swap between x and y:

$$F_2 = xP_x + yP_y - sE + tP_s \tag{5.17}$$

Transverse co-ordinates are unaffected, but

$$p_s = -E \ , \qquad e = -P_s \ , \qquad S = t \ , \qquad T = s \tag{5.18}$$

Call the new (regular phase space) Hamiltonian K, with new extended Hamiltonian $K - E = K + p_s$ which has to equal 0. We find K by solving

$$H\left(x, y, s, p_x, p_y, -K\left(x, y, t, p_x, p_y, -e, s\right), t\right) - e = 0 \ . \tag{5.19}$$

If H is independent of t, then K does as well. The dynamic equations are $e' = -\partial K/\partial t = 0$, $t' = -\partial K/\partial e$.

Example: Lorentz transformation

The generating function for the Lorentz transformation (ignoring transverse coordinates) is

$$F_2(q, t, P, E) = \gamma \left[Pq - Et - v \left(Pt - \frac{Eq}{c^2} \right) \right] \ . \qquad (5.20)$$

Here v is the velocity shift, $\gamma = (1 - v^2/c^2)^{-1/2}$. This yields

$$\begin{aligned} p &= \gamma(P + Ev/c^2) \ , \qquad & e &= \gamma(E + vP) \ , \\ Q &= \gamma(q - vt) \ , \qquad & T &= \gamma(t - vq/c^2) \ . \end{aligned} \qquad (5.21)$$

For consideration

- A damped harmonic oscillator can be described in a way that seems Hamiltonian (example given). Does this make sense?
- How is mechanical momentum a meaningful concept when canonical momentum determines the dynamics?

Coordinate system and Hamiltonian in a circular accelerator (Lecture 6)

January 27, 2016

Lecture outline

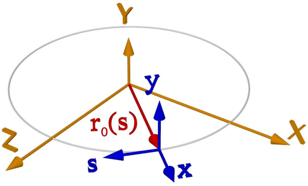
We first introduce a special coordinate system for a circular accelerator and write equations of motion in that system. We then switch to the Hamiltonian in which the coordinate s plays a role of time. Finally, we make a small-amplitude approximation in the Hamiltonian and derive a simplified expression for it.

Setup

We assume that there is no electrostatic fields, $\phi = 0$, and the magnetic field is static. The magnetic field directs particle's motion in such a way that the particle is moving in a closed orbit. This *reference* orbit is established for a particle with the nominal momentum p_0 $(= m\gamma_0 v_0)$. Our goal is to describe particles' motion in the vicinity of this reference orbit, with energies (momenta) that can slightly deviate from the nominal one. We will also assume that the reference orbit is a plane curve.

Reference orbit

A reference orbit and Cartesian coordinate system X, Y and Z.



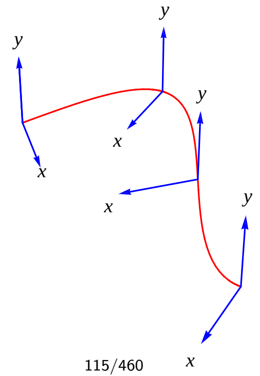
It is given by the vector $r_0(s)$, where s is the arclength measured along the orbit *in the direction of motion*. We define three unit vectors. Vector $\hat{s}$ is the tangential vector to the orbit, $\hat{s} = dr_0/ds$. Vector, $\hat{x}$, is perpendicular to $\hat{s}$ and lies in the plane of the orbit. Vector $\hat{y}$ is $\hat{y} = \hat{s} \times \hat{x}$. It is perpendicular to the plane of the orbit, and hence to $\hat{s}$ and $\hat{x}$

The three vectors $\hat{x}$, $\hat{y}$, and $\hat{s}$ constitute a right-hand oriented base for the local coordinate system. The coordinate x is measured along $\hat{x}$, and the coordinate y is measured along $\hat{y}$.

Reference orbit

Note that simultaneously flipping the directions of vectors x and y is allowable, because it transforms a right-hand oriented coordinate system to another one.

If the direction of the motion is reversed (e.g., by changing the direction of the magnetic field or the sign of charge of the particles), then vector $\hat{s}$ changes direction. To keep the local coordinate system right-handed, the direction of vector $\hat{x}$ is usually reversed too.



Reference orbit

From the differential geometry (the so called Frenet-Serret formulas) we have the following relations between the derivatives of vectors r_0, $\hat{s}$, $\hat{x}$, and $\hat{y}$:

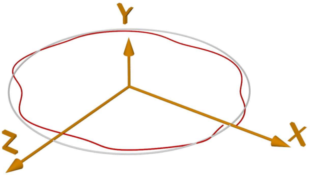
$$\begin{aligned}
\frac{dr_0}{ds} &= \hat{s}\,, \\
\frac{d\hat{s}}{ds} &= -\frac{\hat{x}}{\rho(s)}\,, \\
\frac{d\hat{x}}{ds} &= \frac{\hat{s}}{\rho(s)}\,, \\
\frac{d\hat{y}}{ds} &= 0\,.
\end{aligned} \tag{6.1}$$

Deviations from reference orbit

Since we assumed that the orbit is plane, the magnetic field *on the orbit* can only have y (vertical field) and s (solenoidal field) components. The bending radius ρ is given by the following equation (recall (3.17))

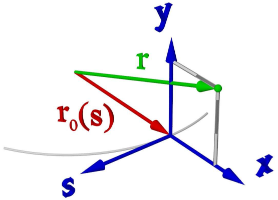
$$\rho(s) = \frac{p_0}{eB_y(s)}\,. \tag{6.2}$$

Most of the particles in the beam deviate from the reference orbit, although they move close to it. Our goal to describe motion of these particles.



Coordinate system

Each point *in the vicinity of the central orbit* can be represented in the local coordinate system.



In this system a radius vector r is represented by coordinates s, x, and y such that

$$r = r_0(s) + x\hat{x}(s) + y\hat{y}\,. \tag{6.3}$$

Derivatives in curvilinear coordinate system

Useful formulae for the gradient of a scalar function $\phi(x, y, s)$, and for the curl and divergence of a vector function $A = (A_x(x, y, s), A_y(x, y, s), A_s(x, y, s))$:

$$\nabla\phi = \hat{x}\frac{\partial\phi}{\partial x} + \hat{y}\frac{\partial\phi}{\partial y} + \hat{s}\frac{1}{1 + x/\rho}\frac{\partial\phi}{\partial s}\,, \tag{6.4}$$

$$(\nabla \times A)_x = -\frac{1}{1 + x/\rho}\frac{\partial A_y}{\partial s} + \frac{\partial A_s}{\partial y}\,, \tag{6.5}$$

$$(\nabla \times A)_s = -\frac{\partial A_x}{\partial y} + \frac{\partial A_y}{\partial x}\,, \tag{6.6}$$

$$(\nabla \times A)_y = -\frac{1}{1 + x/\rho}\frac{\partial A_s(1 + x/\rho)}{\partial x} + \frac{1}{1 + x/\rho}\frac{\partial A_x}{\partial s}\,, \tag{6.7}$$

$$\nabla \cdot A = \frac{1}{1 + x/\rho}\frac{\partial A_x(1 + x/\rho)}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{1}{1 + x/\rho}\frac{\partial A_s}{\partial s}, \tag{6.8}$$

[remember that $\rho = \rho(s)$].

Hamiltonian, canonical transformation to new variables

The Hamiltonian for a charged particle is

$$H = \sqrt{(mc^2)^2 + c^2(\boldsymbol{\pi} - eA)^2}\,. \tag{6.9}$$

This Hamiltonian was derived for a Cartesian coordinate system which coordinates we denote X, Y and Z. We now want to obtain a Hamiltonian in the coordinate system related to the reference orbit. We will use generating functions to transform the Hamiltonian to the new coordinates.

As a first step, we choose local coordinates s, x, and y as coordinate variables of our new Hamiltonian. To carry out a transformation from the original Cartesian coordinates X, Y and Z to the new ones, we will a the generating function of the third type:

$$F_3(\boldsymbol{\pi}, x, y, s) = -\boldsymbol{\pi} \cdot \left(r_0(s) + x\hat{x}(s) + y\hat{y}\right). \tag{6.10}$$

In this equation $\boldsymbol{\pi} = (\pi_X, \pi_Y, \pi_Z)$ is the *old* momentum and x, y and z are the new coordinates.

Hamiltonian, canonical transformation to new variables

We denote by $\boldsymbol{\Pi} = (\Pi_x, \Pi_y, \Pi_s)$ the new canonical momentum (see Eqs. for the generating function F_3)

$$
\begin{aligned}
\Pi_x &= -\frac{\partial F_3}{\partial x} = \boldsymbol{\pi} \cdot \hat{x} = \pi_x \,, \\
\Pi_y &= -\frac{\partial F_3}{\partial y} = \boldsymbol{\pi} \cdot \hat{y} = \pi_y \,, \\
\Pi_s &= -\frac{\partial F_3}{\partial s} = \boldsymbol{\pi} \cdot \left(\frac{d\mathbf{r_0}}{ds} + x \frac{d\hat{x}}{ds} \right) \\
&= \boldsymbol{\pi} \cdot \left(\hat{s} + \frac{x}{\rho} \hat{s} \right) \\
&= \pi_s \left(1 + \frac{x}{\rho} \right) \,.
\end{aligned}
\tag{6.11}
$$

Hamiltonian, canonical transformation to new variables

Note that

$$
\begin{aligned}
(\boldsymbol{\pi} - eA)^2 &= (\pi_x - eA_x)^2 + (\pi_y - eA_y)^2 + (\pi_s - eA_s)^2 \qquad (6.12)\\
&= (\Pi_x - eA_x)^2 + (\Pi_y - eA_y)^2 + \left(\frac{\Pi_s}{1 + x/\rho} - eA_s\right)^2 ,
\end{aligned}
$$

and our Hamiltonian becomes

$$
H = c\left[m^2c^2 + (\Pi_x - eA_x)^2 + (\Pi_y - eA_y)^2 + \left(\frac{\Pi_s}{1 + x/\rho} - eA_s\right)^2\right]^{1/2} . \qquad (6.13)
$$

We have used the notation $A_x = A \cdot \hat{x}$, $A_y = A \cdot \hat{y}$, and $A_s = A \cdot \hat{s}$.
[Some authors use $A_s = (1 + x/\rho)A \cdot \hat{s}$.]
Eq. (6.13) is our new Hamiltonian as a function of new coordinates x, y, s and new conjugate momenta Π_x, Π_y and Π_s.

Using s as a time variable

A particle has three degrees of freedom with the Hamiltonian which is a constant of motion because it does not depend on time t. Using this fact we can lower the number of degrees of freedom from 3 to 2. To do this, we choose the variable s as an independent variable instead of the time t.
Let us assume that we solved equations of motion and found all the variables as functions of time, $x(t)$, $y(t)$, $s(t)$, etc. Then we invert the expression $s = s(t)$ to find $t(s)$, and $x(t) \to x(t(s)) \equiv x(s)$. We can do the same trick with coordinate y and components of the momentum $\mathbf{\Pi}$, and define $y(s)$ and $\mathbf{\Pi}(s)$.

In accelerators beam diagnostics are located at given s, so it is often natural to work with $x(s)$ and $y(s)$ rather than with $x(t)$ and $y(t)$.

Using s as a time variable

It turns out that the dependence of x, y, Π_x, and Π_y versus s, can be found from Hamiltonian equations of motion, using a new Hamiltonian with 2 pairs of conjugate variables. We first formulate how to calculate this new Hamiltonian, and then prove that using the formulated approach we indeed obtain the new equations of motion for the two pairs of the canonically conjugate variables. Let us write down the following equation:

$$h = H(x, \Pi_x, y, \Pi_y, s, \Pi_s)\,, \tag{6.14}$$

(where H is given by (6.13)) and solve it for Π_s

$$\Pi_s = \Pi_s(x, \Pi_x, y, \Pi_y, s, h)\,. \tag{6.15}$$

Here h is the value of the Hamiltonian H. Because our Hamiltonian does not depend on time, the value of H is constant along each orbit, the value of the Hamiltonian is equal to γmc^2.

New Hamiltonian

Let us introduce now a new Hamiltonian K

$$K(x, \Pi_x, y, \Pi_y, s, h) = -\Pi_s(x, \Pi_x, y, \Pi_y, s, h)\,, \tag{6.16}$$

in which x, Π_x, y, Π_y are canonical conjugate variables, s is an independent "time" variable, and h is a (constant) parameter. The Hamiltonian K has two pairs of conjugate variables, and hence describes motion of a system which has two degrees of freedom. However, this Hamiltonian depends on "time" s.

We now show that dependence $x(s)$, $\Pi_x(s)$, $y(s)$, and $\Pi_y(s)$ are governed by the Hamiltonian (6.16). We have to remember that, e.g., $x(s)$ is obtained from $x(t)$ and $s(t)$ by eliminating the variable t, and $x(t)$ with $s(t)$ are governed by the original Hamiltonian H. We have for dx/ds

$$\frac{dx}{ds} = \frac{dx/dt}{ds/dt} = \frac{\partial H/\partial \Pi_x}{\partial H/\partial \Pi_s}\,. \tag{6.17}$$

New Hamiltonian

On the other hand, the derivative $\partial K/\partial \Pi_x$ can be calculated as a derivative of an implicit function

$$\frac{\partial K}{\partial \Pi_x} = -\left(\frac{\partial \Pi_s}{\partial \Pi_x}\right)_H = \frac{\partial H/\partial \Pi_x}{\partial H/\partial \Pi_s}\,, \tag{6.18}$$

and we see that

$$\frac{dx}{ds} = \frac{\partial K}{\partial \Pi_x}\,. \tag{6.19}$$

The same approach works for Π_x,

$$\frac{d\Pi_x}{ds} = \frac{d\Pi_x/dt}{ds/dt} = \frac{-\partial H/\partial x}{\partial H/\partial \Pi_s} = \left(\frac{\partial \Pi_s}{\partial x}\right)_H = -\frac{\partial K}{\partial x}\,. \tag{6.20}$$

New Hamiltonian

Similarly one can show that equations for y and Π_y can be obtained with the Hamiltonian K. The price for lowering the number of degrees of freedom is that we now have a "time dependent" Hamiltonian (K is a function of s).

Although time is eliminated from our equations, the time dependence can be found, if necessary. In order to do this, we need the function $s(t)$. Since $ds/dt = \partial H/\partial \Pi_s$, the inverse function $t(s)$ satisfies the following equation

$$\frac{dt}{ds} = \frac{\partial \Pi_s}{\partial H} = -\frac{\partial K}{\partial h}\,. \tag{6.21}$$

Integrating this equation, we can find $t(s)$, invert it, and find $s(t)$.

Note: even for H time-independent, K has explicit s-dependence. Equations of motion for t and h will be trivial.

Small amplitude approximation

The Hamiltonian K given by Eq. (6.16) can easily be found from Eq. (6.13):

$$K = -\left(1 + \frac{x}{\rho}\right)\left[\frac{1}{c^2}h^2 - (\Pi_x - eA_x)^2 - (\Pi_y - eA_y)^2 - m^2c^2\right]^{1/2} - eA_s\left(1 + \frac{x}{\rho}\right). \tag{6.22}$$

In many cases of interest, a single component A_s is sufficient to describe the magnetic field, so we can set $A_x = A_y = 0$ in Eq. (6.22). In this case, Π_x and Π_y are equal to the kinetic momenta $\Pi_x = p_x = m\gamma v_x$ and $\Pi_y = p_y = m\gamma v_y$ (see Eqs. (6.11) and (3.25)). We will use p_x and p_y instead of Π_x and Π_y in what follows. We will consider these momenta as small quantities (compared with the total momentum of the particle), because particles usually move at a small angle to the nominal orbit.

Small amplitude approximation

Expanding the Hamiltonian in p_x and p_y we get:

$$K \approx -p\left(1+\frac{x}{\rho}\right)\left(1-\frac{p_x^2}{2p^2}-\frac{p_y^2}{2p^2}\right)-eA_s\left(1+\frac{x}{\rho}\right), \quad (6.23)$$

where $p = \sqrt{h^2/c^2 - m^2c^2}$ is the kinetic momentum of the particle (which together with the energy is a conserved quantity in a constant magnetic field).

Instead of using dimensional momenta p_x and p_y it is convenient to introduce dimensionless variables $P_x = p_x/p_0$ and $P_y = p_y/p_0$, where p_0 is the nominal momentum in the ring. Transformation from x, p_x, y, p_y to x, P_x, y, P_y is not canonical, but a simple consideration shows that it can be achieved by simply dividing the Hamiltonian by p_0.

Small amplitude approximation

Denoting the new Hamiltonian $\mathcal{H}$ we have

$$\mathcal{H}(x, P_x, y, P_y) = \frac{K}{p_0} \tag{6.24}$$

$$= -\frac{p}{p_0}\left(1+\frac{x}{\rho}\right)\left(1-\frac{1}{2}P_x^2\left(\frac{p}{p_0}\right)^2-\frac{1}{2}P_y^2\left(\frac{p}{p_0}\right)^2\right)-\frac{e}{p_0}A_s\left(1+\frac{x}{\rho}\right)$$

We are interested here in the case when the energy and the total momentum of the particle can only slightly deviate from the nominal one, that is

$$\frac{p}{p_0} = 1+\eta, \tag{6.25}$$

with $\eta \ll 1$. With this in mind, we obtain

Small amplitude approximation

$$\mathcal{H}(x, P_x, y, P_y) \tag{6.26}$$
$$= -(1+\eta)\left(1+\frac{x}{\rho}\right)\left(1-\frac{1}{2}P_x^2-\frac{1}{2}P_y^2\right)-\frac{e}{p_0}A_s\left(1+\frac{x}{\rho}\right),$$

where we replaced $(p/p_0)^2$ by unity in small quadratic terms proportional to P_x^2 and P_y^2.
Finally, we note that our momenta P_x, P_y are approximately equal to the orbit slopes $x' \equiv dx/ds$ and $y' \equiv dy/ds$, respectively. Indeed

$$x' \equiv \frac{dx}{ds} = \frac{v_x}{v_s} = \frac{p_x}{p_s} \approx P_x\,, \tag{6.27}$$

with a similar expression for y'. Some authors actually use x' and y' as canonical momenta conjugate to x and y instead of P_x, P_y—in this case one has to be careful to avoid confusion between a canonical variable (say P_x) with the rate of change of its conjugate (that is dx/ds).

Time dependent Hamiltonian

Our derivation above can be easily modified to include the case of time dependent Hamiltonians. In Eq. (6.14) we will have $H(x, \Pi_x, y, \Pi_y, s, \Pi_s, t)$, and correspondingly the new Hamiltonian K will also be a function of time

$$K(x, \Pi_x, y, \Pi_y, t, h, s) = -\Pi_s(x, \Pi_x, y, \Pi_y, t, h, s), \qquad (6.28)$$

where the time t is now understood as a third coordinate (in addition to x and y) and the energy h is the third momentum. The Hamiltonian equation (6.21) should be complemented by

$$\frac{dh}{ds} = \frac{\partial K}{\partial t}. \qquad (6.29)$$

Note that Eqs (6.21) and (6.29) have "inverted" signs if t is treated as a coordinate and h as it conjugate momentum. This, however, can be easily fixed, if one accepts $-h$ as the momentum conjugate to t.

Equations of motion in an accelerator (Lecture 7)

January 27, 2016

Lecture outline

We consider several types of magnets used in accelerators and write down the vector potential of the magnetic field for them. We then analyze small betatron oscillations in the ring.

Vector potential for different types of magnets

There are several types of magnets that are used in accelerators and each of them is characterized by a specific dependence of A_s versus x and y. We will use the fact that we are only interested in fields near the reference orbit, $|x|, |y| \ll |\rho|$. We will be neglecting higher order terms such as $(x/\rho)^2$ and $(y/\rho)^2$.

We first consider *dipole* magnets that are used to bend the orbit. The dipole magnetic field is:

$$B = \hat{y} B(s)\,. \tag{7.1}$$

The function $B(s)$ is such that it is not zero only inside the magnet and vanishes outside of it. This field can be represented by the following vector potential:

$$A_s = -B(s) x \left(1 - \frac{x}{2\rho}\right)\,. \tag{7.2}$$

Vector potential for different types of magnets

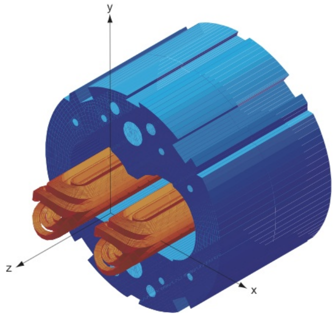
Indeed, using Eq. (6.4)

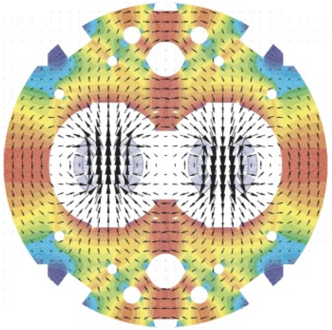
$$
\begin{aligned}
B_y &= -\frac{1}{1+x/\rho}\frac{\partial A_s(1+x/\rho)}{\partial x} \\
&\approx B(s)\left(1-\frac{x}{\rho}\right)\frac{\partial}{\partial x}x\left(1-\frac{x}{2\rho}\right)\left(1+\frac{x}{\rho}\right) \\
&\approx B(s)\left(1-\frac{x}{\rho}\right)\frac{\partial}{\partial x}\left(x+\frac{x^2}{2\rho}\right) \\
&\approx B(s)+O\left(\frac{x^2}{\rho^2}\right)\ . \qquad (7.3)
\end{aligned}
$$

This is an approximation in which we only keep terms to the first order in $|x/\rho|$.

Vector potential for different types of magnets

The picture of windings of a dipole magnet.



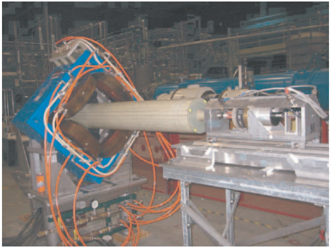


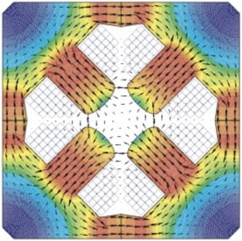
Vector potential for different types of magnets

The second type is a *quadrupole* magnet. It is used to focus off-orbit particles close to the reference orbit. It has the following magnetic field:

$$B = G(s)(\hat{y}x + \hat{x}y)\,. \tag{7.4}$$

The picture of a quadrupole magnet and the magnetic field lines.





Vector potential for different types of magnets

As we will see below from the equations of motion, the quadrupole magnetic field focuses particles around the equilibrium orbit. The corresponding vector potential is

$$A_s = \frac{G}{2}\left(y^2 - x^2\right) . \tag{7.5}$$

A *skew quadrupole* is a normal quadrupole rotated by 45 degrees:

$$B = G_s(s)(-\hat{y}y + \hat{x}x) , \tag{7.6}$$

with

$$A_s = G_s xy . \tag{7.7}$$

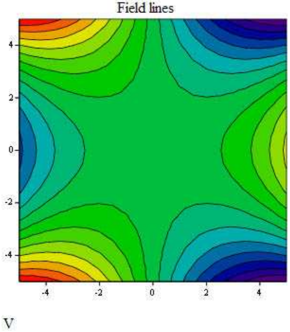
Vector potential for different types of magnets

Finally, we will also consider a *sextupole magnet*. Sextupoles are used to correct some properties of betatron oscillations of particles around the reference orbit. This element has a nonlinear dependence of the magnetic field with transverse coordinates:

$$B = S(s)\left[\frac{1}{2}\hat{y}(x^2 - y^2) + \hat{x}xy\right] , \tag{7.8}$$

with

$$A_s = S\left(\frac{1}{2}xy^2 - \frac{1}{6}x^3\right) . \tag{7.9}$$



Linearized Hamiltonian

Assume that we have only dipoles and quadrupoles in the ring:

$$
\begin{aligned}
\mathcal{H} &\approx -(1+\eta)\left(1+\frac{x}{\rho}\right)\left(1-\frac{1}{2}P_x^2-\frac{1}{2}P_y^2\right)-\frac{e}{p_0}A_s\left(1+\frac{x}{\rho}\right) \\
&= -(1+\eta)\left(1+\frac{x}{\rho}\right)\left(1-\frac{1}{2}P_x^2-\frac{1}{2}P_y^2\right) \\
&-\frac{e}{p_0}\left[-B(s)x\left(1-\frac{x}{2\rho}\right)+\frac{G(s)}{2}\left(y^2-x^2\right)\right]\left(1+\frac{x}{\rho}\right) \\
&\approx -1-\eta-\eta\frac{x}{\rho}+\frac{1}{2}P_x^2+\frac{1}{2}P_y^2+\frac{x^2}{2\rho^2}-\frac{e}{p_0}\frac{G(s)}{2}\left(y^2-x^2\right)\,,
\end{aligned}
\tag{7.10}
$$

where we use $\rho = p/eB$ and neglected terms of the third and higher orders. Assume $\eta = 0$. We can drop the constant first term (unity) on the last line of Eq. (7.10), then the Hamiltonian is the sum of two terms corresponding to vertical and horizontal degrees of freedom:

$$
\mathcal{H} = \mathcal{H}_x + \mathcal{H}_y\,, \tag{7.11}
$$

More generally, the motion can be coupled which is more complicated.

Linearized Hamiltonian

with

$$\mathcal{H}_x = \frac{1}{2}P_x^2 + \frac{x^2}{2\rho^2} + \frac{e}{p_0}\frac{G(s)}{2}x^2\,, \tag{7.12}$$

and

$$\mathcal{H}_y = \frac{1}{2}P_y^2 - \frac{e}{p_0}\frac{G(s)}{2}y^2\,. \tag{7.13}$$

The Hamiltonian (7.11) is split into a sum of two Hamiltonians—the horizontal and vertical motions are decoupled. The quadrupoles focus or defocus the beam in the transverse direction: focusing in x ($G > 0$) results in defocusing in y, and vice versa. We will show that a sequence of quadrupoles with alternating polarities confine the beam near the reference orbit. A particle near the equilibrium orbit executes *betatron* oscillations. So does the beam centroid, though dynamics may be different.

Linearized Hamiltonian

Notice also focusing in the horizontal direction inside dipole magnets (so-called *weak* focusing $\propto 1/\rho^2$).
In what follows, to study general properties of the transverse motion in both transverse planes, we will use a generic Hamiltonian

$$\mathcal{H}_0 = \frac{1}{2}P_x^2 + \frac{K(s)}{2}x^2\,, \tag{7.14}$$

where $K = \rho^{-2} + eG/2p_0$ for the horizontal, and $K = -eG/2p_0$ for the vertical plane.

Hill's equation

From the Hamiltonian (7.14) we find the following equation of motion in a transverse plane:

$$x''(s) + K(s)x(s) = 0\,. \tag{7.15}$$

In an accelerator ring $K(s)$ is a *periodic* function of s with a period that we denote by L (which may be equal to a fraction of the ring circumference), and Eq. (7.15) is called Hill's equation. It describes the so called *betatron* oscillations of a particle in the ring. Note that the same equation describes the parametric resonance, with the only difference that we now have s as an independent variable instead of t. We now know that this equation can have both stable and unstable solutions. Of course, for storage and acceleration of beams in an accelerator, one has to design it in such a way that avoids unstable solutions of Eq. (7.15).

Betatron function

To understand general properties of the betatron motion in a ring, we seek a solution to Eq. (7.15) in the following form

$$x(s) = Aw(s)\cos\psi(s)\,, \tag{7.16}$$

where A is an arbitrary constant, and ψ is called the *betatron phase*. It turns out that if a particle's motion is stable, $w(s)$ is a periodic function of s with the period L. Introducing two unknown functions $w(s)$ and $\psi(s)$ instead of one function $x(s)$ gives us freedom to impose a constraint later in the derivation. We have

$$\begin{aligned}
\frac{x'}{A} &= w'\cos\psi - w\psi'\sin\psi\,, \\
\frac{x''}{A} &= w''\cos\psi - 2w'\psi'\sin\psi - w\psi''\sin\psi - w\psi'^2\cos\psi\,.
\end{aligned} \tag{7.17}$$

Betatron function

Eq. (7.15) now becomes

$$\begin{aligned} & w'' \cos\psi - 2w'\psi' \sin\psi - w\psi'' \sin\psi - w\psi'^2 \cos\psi + K(s) w \cos\psi \\ & = [w'' \cos - w\psi'^2 + K(s) w] \cos\psi - [2w'\psi' + w\psi''] \sin\psi = 0 \,. \end{aligned} \tag{7.18}$$

We now use the freedom mentioned above and set to zero both the term in front of $\cos\psi$ and the term in front of $\sin\psi$. This gives us two equations:

$$\begin{aligned} w'' - w\psi'^2 + K(s) w &= 0 \\ -2w'\psi' - w\psi'' &= 0 \,. \end{aligned} \tag{7.19}$$

Betatron function

The last equation can be written as

$$\frac{1}{w}(\psi' w^2)' = 0\,. \tag{7.20}$$

If we introduce the β *function* as $\beta(s) = w(s)^2$, then

$$\psi' = \frac{1}{\beta(s)} \tag{7.21}$$

(we have chosen the constant of integration equal to 1). The first equation in (7.19) now becomes

$$w'' - \frac{1}{w^3} + K(s)w = 0\,, \tag{7.22}$$

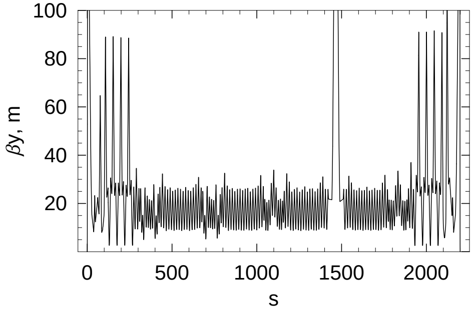
or, in terms of the β function,

$$\frac{1}{2}\beta\beta'' - \frac{1}{4}\beta'^2 + K\beta^2 = 1\,. \tag{7.23}$$

This is a nonlinear differential equation of second order. As we pointed out above, one has to find a periodic solution to this equation.

Betatron function

There are several techniques and codes to find $\beta(s)$ and other orbit properties for a ring with given magnets. Transfer maps from element to element are often used as a starting point: there is a direct relation between a symplectic transfer map and how α and β change (also phase advance or tune).
Here is the beta function for the High Energy Ring of PEP-II at SLAC.



Tune

The betatron phase advance of the ring can be found by integrating Eq. (7.21),

$$\Delta\psi = \int_0^C \frac{ds}{\beta(s)} \,. \tag{7.24}$$

The quantity $\Delta\psi/2\pi$ is called the *tune* ν (in European literature it is usually denoted by Q)

$$\nu = \frac{1}{2\pi} \int_0^C \frac{ds}{\beta(s)} \,; \tag{7.25}$$

it is a fundamental characteristic of the beam dynamics in the ring.

Action-angle variables for circular machines (Lecture 8)

January 27, 2016

Action-angle variables and the Floquet transformation

We start with the Hamiltonian

$$\mathcal{H} = \frac{1}{2}P_x^2 + \frac{1}{2}K(s)x^2 \,.$$

Previously we found a solution to this equation which we now write as

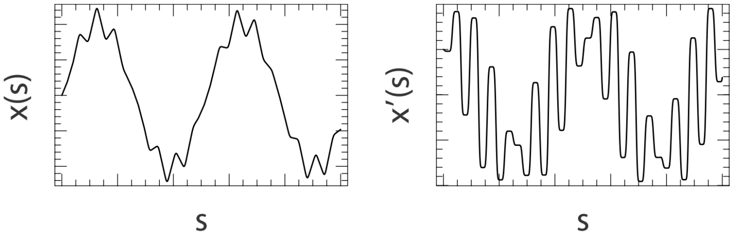
$$x(s) = A\sqrt{\beta(s)}\cos\psi(s)\,. \tag{8.1}$$

Differentiating this equation with respect to s we find

$$\begin{aligned} x'(s) = P_x &= A\frac{\beta'}{2\sqrt{\beta(s)}}\cos\psi(s) - \sqrt{\beta(s)}\psi'\sin\psi(s) \\ &= \frac{A}{\sqrt{\beta}}\cos\psi(s)\left(\frac{\beta'}{2} - \tan\psi(s)\right) \\ &= \frac{x}{\beta}\left(\frac{\beta'}{2} - \tan\psi(s)\right)\,. \end{aligned} \tag{8.2}$$

Action-angle variables and the Floquet transformation

Typically x and x' look like functions shown here.



We will now show that transforming to new variables, we can obtain a much simpler description of the particle motion.

Action-angle variables and the Floquet transformation

We will obtain the action-angle variables ϕ (coordinate) and J (momentum) using a generating function of the first kind, (4.22), $F_1(x, \phi, s)$. Analogous to the canonical transformation for the linear oscillator we will require that

$$x(s) = A(J)\sqrt{\beta(s)}\cos\phi$$
$$P_x(s) = \frac{x}{\beta(s)}\left(\frac{\beta'}{2} - \tan\phi\right) . \tag{8.3}$$

We have

$$P_x = \frac{\partial F_1}{\partial x} \qquad J = -\frac{\partial F_1}{\partial \phi}$$

The generating function is

$$F_1(x, \phi, s) = \int P_x dx = \frac{x^2}{2\beta}\left(\frac{\beta'}{2} - \tan\phi\right) , \tag{8.4}$$

where for P_x we used Eq. (8.3).

Action-angle variables and the Floquet transformation

With this generating function we find the action

$$J = -\frac{\partial F_1}{\partial \phi} = \frac{x^2}{2\beta} \sec^2 \phi \, . \tag{8.5}$$

Using $\sec^2 \phi = 1 + \tan^2 \phi$, and the expression for $\tan \phi$ from Eq. (8.3),

$$-\tan \phi = \frac{\beta P_x}{x} + \alpha \tag{8.6}$$

where

$$\alpha = -\frac{\beta'}{2} \, , \tag{8.7}$$

we obtain J in terms of x and x':

$$J = \frac{1}{2\beta} \left[x^2 + (\beta P_x + \alpha x)^2 \right] \, . \tag{8.8}$$

Action-angle variables and the Floquet transformation

Equations (8.6) and (8.8) give us the transformation from the old conjugate variables x and P_x to the new ones ϕ and J. The inverse transformation $(\phi, J) \rightarrow (x, P_x)$ can also be found. From the first of Eqs. (8.5) we have

$$x = \sqrt{2\beta J} \cos\phi \,. \tag{8.9}$$

Substituting this relation to the second of Eqs. (8.5) we obtain the equation for P_x in terms of J and ψ

$$P_x = -\sqrt{\frac{2J}{\beta}} \left(\sin\phi + \alpha\cos\phi\right) \,. \tag{8.10}$$

Action-angle variables and the Floquet transformation

To find the new Hamiltonian which we denote by $\hat{\mathcal{H}}$ we need to take into account that the generating function depends on the time-like variable s:

$$
\begin{aligned}
\hat{\mathcal{H}} &= \mathcal{H} + \frac{\partial F_1}{\partial s} \\
&= \frac{1}{2} P_x^2 + \frac{1}{2} K(s) x^2 + \frac{\partial}{\partial s} \frac{x^2}{2\beta} \left(\frac{\beta'}{2} - \tan\phi \right) \\
&= \frac{1}{2} P_x^2 + \frac{1}{2} K(s) x^2 + \frac{x^2}{4} \frac{\beta''\beta - \beta'^2}{\beta^2} + \frac{x^2 \beta'}{2\beta^2} \tan\phi \,. \qquad (8.11)
\end{aligned}
$$

Action-angle variables and the Floquet transformation

We now use

$$\frac{1}{2}\beta\beta'' - \frac{1}{4}\beta'^2 + K\beta^2 = 1\,.$$

to eliminate β'' from the equation and Eq. (8.6) to eliminate ϕ:

$$\begin{aligned}
\hat{\mathcal{H}} &= \\
&= \frac{1}{2}P_x^2 + \frac{1}{2}K(s)x^2 - \frac{x^2}{4}\frac{\frac{1}{2}\beta'^2 + 2K\beta^2 - 2}{\beta^2} - \frac{x^2\beta'}{2\beta^2}\left(\frac{\beta x'}{x} + \alpha\right) \\
&= \frac{1}{2}P_x^2 + \frac{1}{2\beta^2}x^2 + \frac{\alpha^2}{2\beta^2}x^2 + \frac{\alpha}{\beta}P_x x \\
&= \frac{1}{2\beta^2}x^2 + \frac{1}{2}\left(P_x + \frac{\alpha}{\beta}x\right)^2 \\
&= \frac{J}{\beta}\,.
\end{aligned} \tag{8.12}$$

Action-angle variables and the Floquet transformation

Since the new Hamiltonian is independent of ϕ the equation for J is

$$J' = \frac{\partial \hat{\mathcal{H}}}{\partial \phi} = 0\,, \tag{8.13}$$

which means that J is an integral of motion. The quantity $2J$ is called the Courant-Snyder invariant. The Hamiltonian equation for ϕ gives

$$\phi' = \frac{\partial \hat{\mathcal{H}}}{\partial J} = \frac{1}{\beta(s)}\,. \tag{8.14}$$

Comparing this equation with Eq. (7.21) we see that the new variable ϕ is actually the old betatron phase, $\phi = \psi + \phi_0$. Particles in the beam will have various initial phases ϕ_0.

Action-angle variables and the Floquet transformation

We "straightened out" the behavior of the new momentum variable.

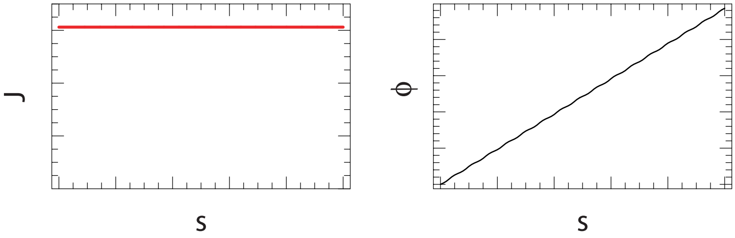


Figure : Plots of J and ϕ versus s.

Action-angle variables and the Floquet transformation

We can go further and "straighten out" the ϕ variable as well. This is achieved with one more canonical transformation, from ϕ and J to ϕ_1 and J_1. The generating function is of the second type, $F_2(\phi, J_1, s)$

$$F_2(\phi, J_1, s) = J_1 \left(\frac{2\pi\nu s}{C} - \int_0^s \frac{ds'}{\beta(s')} \right) + \phi J_1 \,, \tag{8.15}$$

where C is the circumference of the ring, and the tune is given by Eq. (7.25). This function gives for the new angle

$$\begin{aligned} \phi_1 = \frac{\partial F_2}{\partial J_1} &= \phi + \frac{2\pi\nu s}{C} - \int_0^s \frac{ds'}{\beta(s')} \\ &= \phi + \frac{2\pi\nu s}{C} - \psi(s) \,, \end{aligned} \tag{8.16}$$

and the action is not changed

$$J = \frac{\partial F_2}{\partial \phi} = J_1 \,. \tag{8.17}$$

Action-angle variables and the Floquet transformation

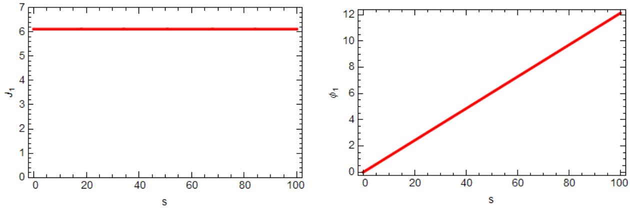
The new Hamiltonian is

$$\hat{\mathcal{H}}_1 = \hat{\mathcal{H}} + \frac{\partial F_2}{\partial s} = \frac{2\pi\nu}{C} J_1 = \mathrm{const}\,. \tag{8.18}$$

Now the evolution of the new coordinate ϕ_1 is governed by the equation

$$\phi_1' = \frac{\partial \hat{\mathcal{H}}_1}{\partial J_1} = \frac{2\pi\nu}{C}\,, \tag{8.19}$$

which means that ϕ_1 is a linear function of s.

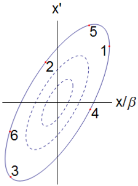


Action-angle variables and the Floquet transformation

Further comments:
This transformation can be useful when analyzing nonlinear dynamics in a ring. Note that $\phi_1{}' \neq 1/\beta$ any more.
We could have chosen instead to change s to phase advance, through an extended canonical transformation.

Phase space motion at a given location

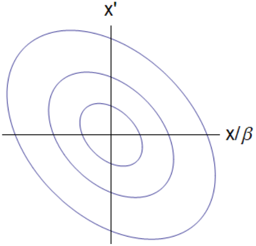
Let us assume that we plot the phase space x, P_x at some location s at the ring and follow particle's motion as it passes through this location. It is convenient to normalize the coordinate x by the beta function at this location $\beta(s)$. A set of consecutive points $x_n/\beta, P_{x,n}$, $n = 1, 2 \ldots$, in the phase space will form particle's trajectory. Because we have an integral of motion J, all these points are located on the curve $J = \mathrm{const}$. From the expression (8.8) for J it follows that this curve is an ellipse whose size and orientation depends on the values J, β, and α.



Ellipses at different positions in the ring

It is easy to see that the ellipse becomes a circle if $\alpha = 0$. In this case, the trajectory is very simple: each consecutive point of the circle is rotated by the betatron phase advance $\Delta\psi$ in the clockwise direction.

Set of ellipses at another location in the ring will have a different shapes which are defined by the local values of β and α.



Imagine how these ellipses are rotating and changing their shape when one travels along the ring circumference.

Dipole field errors and the closed orbit distortion (Lecture 9)

January 28, 2016

Lecture outline

The magnetic field in any real machine is different from the ideal design one. It is important to understand what is the effect of small magnetic errors on particles' motion in an accelerator. In this lecture we consider the effect of the dipole field errors.

Closed orbit distortion

We consider what happens if a dipole magnetic field is not exactly equal to the design one. We will see that such errors lead to changes in the reference orbit in an accelerator.
Let us assume the guiding vertical magnetic field in a circular accelerator deviates from the design value by $\Delta B(s)$. The corresponding vector potential, in which we keep only the first order term [see Eq. (7.2)] is $A_s = -\Delta B(s)x$. This vector potential should be added to the Hamiltonian (7.10); it modifies the motion in the horizontal plane only. We can write $\mathcal{H}_x$ as

$$\mathcal{H} = \frac{1}{2}P_x^2 + \frac{1}{2}K(s)x^2 + \frac{e\Delta B(s)}{p_0}x \tag{9.1}$$

(we drop the subscript x in what follows).

Closed orbit distortion

The most direct way to deal with this problem is to write the equation for x

$$x'' + K(s)x = -\frac{e\Delta B(s)}{p_0}\,. \tag{9.2}$$

A periodic solution, $x_0(s)$, to this equation gives a *closed orbit distortion*. It satisfies Eq. (9.2),

$$x_0'' + K(s)x_0 = -\frac{e\Delta B(s)}{p_0}\,, \tag{9.3}$$

with the periodicity condition $x_0(s + C) = x_0(s)$ where C is the circumference of the ring. A general solution to Eq. (9.2) is

$$x(s) = x_0(s) + \xi(s)\,, \tag{9.4}$$

where $\xi(s)$ satisfies

$$\xi'' + K(s)\xi = 0\,. \tag{9.5}$$

Closed orbit distortion

The function $\xi(s)$ describes betatron oscillations around the perturbed orbit.

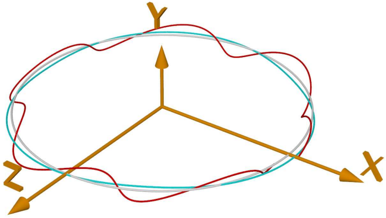


Figure : An ideal and distorted orbits, and a betatron oscillation.

Closed orbit distortion

Let us calculate the orbit distortion $x_0(s)$. We first consider the case of a field perturbation localized at one point:

$$\Delta B(s) = \Delta B_0(s')\delta(s - s') \,. \tag{9.6}$$

Since the right hand side of Eq. (9.3) is equal to zero everywhere except for the point $s = s'$, we seek solution in the form of Eq. (8.1) where we introduce an initial phase ψ_0

$$x_0(s) = A\sqrt{\beta(s)}\cos(\psi(s) - \psi_0) \,. \tag{9.7}$$

Our first requirement is that $x_0(s)$ should be continuous at $s = s'$. This is achieved if we choose $\psi_0 = \psi(s') + \pi\nu$ and assume that $\psi(s)$ varies from $\psi(s')$ to $\psi(s') + 2\pi\nu$ when we go around the orbit. Thus, at s' we have $x_0(s') = A\sqrt{\beta(s')}\cos(-\pi\nu)$. After a turn around the ring, the argument of the cos function becomes equal to $\pi\nu$, and since cos is an even function, $x_0(s' + C) = x_0(s')$.

Closed orbit distortion

The second requirement is obtained by integrating through the δ-function in Eq. (9.3)—it gives us the jump of the derivative of x_0 at s'

$$x_0'(s') - x_0'(s' + C) = -\frac{e\Delta B_0(s')}{p_0}\,. \tag{9.8}$$

From this equation we find

$$A = \frac{\sqrt{\beta(s')}}{2\sin(\pi\nu)}\frac{e\Delta B_0(s')}{p_0}\,. \tag{9.9}$$

For an arbitrary function $B(s)$ we need to add contributions from all locations, which means integration over the circumference of the ring:

$$x_0(s) = \frac{e}{2p_0\sin(\pi\nu)}\int_s^{s+C} ds'\Delta B(s')\sqrt{\beta(s)\beta(s')}\cos(\psi(s) - \psi(s') + \pi\nu)\,. \tag{9.10}$$

Closed orbit distortion

From Eq. (9.10), we see that integer values for the tune ν are not acceptable because they would result in unstable closed orbit.
The action becomes

$$J(x, P_x, s) = \frac{1}{2\beta}\left[(x-x_0)^2 + \left(\beta[P_x - x_0'] + \alpha[x - x_0]\right)^2\right], \quad (9.11)$$

and the Hamiltonian again takes the form J/β. This is one of the homework problems, using the generating function

$$F_1(x, \phi, s) = \frac{[x - x_0(s)]^2}{2\beta}\left(\frac{\beta'}{2} - \tan\phi\right) + x x_0'(s). \qquad (9.12)$$

A crude estimate for x_0 is

$$x_0 \sim \frac{\Delta B}{B} C\beta\rho \sim \beta\frac{\Delta B}{B} \ . \qquad (9.13)$$

Stronger focusing reduces the impact of field errors.

Effect of energy deviation

If particle's energy is not exactly equal to the nominal one, its equilibrium orbit in the horizontal plane changes. It is easy to find the new orbit corresponding to the energy deviation η. From the Hamiltonian (7.10) we see that the extra term due to non vanishing η that involves the coordinate x is $-\eta x/\rho$. Hence instead of (9.1) one gets

$$\mathcal{H} = \frac{1}{2}P_x^2 + \frac{1}{2}K(s)x^2 - \frac{\eta}{\rho}x, \tag{9.14}$$

which is formally obtained from (9.1) by replacing

$$\Delta B \to -\frac{\eta p_0}{e\rho}. \tag{9.15}$$

Effect of energy deviation

Using (9.10) we immediately find that the new orbit is given by

$$x_0(s) = D(s)\eta, \tag{9.16}$$

with the function D

$$D(s) = \frac{\sqrt{\beta(s)}}{2\sin(\pi\nu)} \int_s^{s+C} ds' \frac{\sqrt{\beta(s')}}{\rho(s')} \cos(\psi(s) - \psi(s') - \pi\nu)\,. \tag{9.17}$$

This function is called the *dispersion function* of the ring.
Using the expression (9.11) and (9.16) one immediately concludes that the action variable for a particle with energy deviation η is

$$\begin{aligned} J(x, P_x, \eta, s) = \frac{1}{2\beta}\Big[&(x - \eta D(s))^2 + \\ &\big(\beta[P_x - \eta D'(s)] + \alpha[x - \eta D(s)]\big)^2\Big]\,. \end{aligned} \tag{9.18}$$

Quadrupole errors

Let us now assume that we have a quadrupole error

$$\mathcal{H} = \frac{1}{2}x'^2 + \frac{1}{2}K(s)x^2 + \frac{1}{2}\epsilon\Delta K(s)x^2\,, \tag{9.19}$$

where we introduced a formal smallness parameter ϵ. What kind of effects does this error have on the motion? Since we know that the focusing function $K(s)$ determines the betatron oscillations in the system, changing the focusing would result in a perturbation of the beta function and hence the tune of the ring.

Quadrupole errors

Skipping the derivation, I will show the final result.

$$\beta_1 = \beta - \frac{\beta}{2\sin 2\pi\nu}\int_s^{s+C} ds' \Delta K(s')\beta(s')\cos 2(-\psi(s) + \psi(s') - \pi\nu)\,.$$

The last term is often called the *beta beating* term.
An important conclusion that follows from the above equation is that one should avoid half-integer values of the tune—they are unstable with respect to errors in the focusing strength of the lattice.

Third-order resonance

Effect of sextupoles on betatron oscillations. The sextupole vector potential is given by Eq. (7.9) [see Lecture 7]:

$$A_s = S\left(\frac{1}{2}xy^2 - \frac{1}{6}x^3\right). \tag{9.20}$$

Our goal is to study 1D effects so we neglect the first term in this equation (assuming $y = 0$) and use $A_s = -S(s)x^3/6$. We need to add the term $-eA_s/p$ to the Hamiltonian (7.11)

$$\mathcal{H} = \frac{1}{2}x'^2 + \frac{1}{2}K(s)x^2 + \frac{1}{6}\mathcal{S}(s)x^3, \tag{9.21}$$

where $\mathcal{S} = eS/p$.

Third-order resonance

We consider a ring with one short sextupole magnet with length much shorter then the ring circumference C. In this case $\mathcal{S}(s)$ can be approximated by a delta function, $\theta = 2\pi s/C$

$$\mathcal{S}(s) = \mathcal{T}_0 \delta(s - s_0), \qquad R = \frac{1}{2\sqrt{2}} \mathcal{T}_0 \beta_0^{3/2}$$

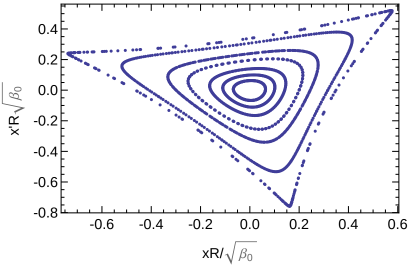
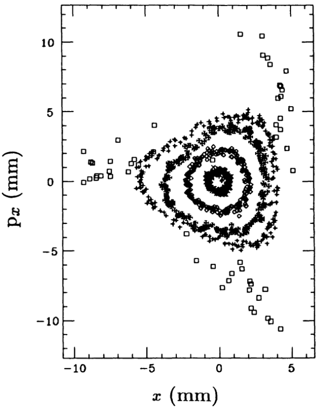


Figure : Phase orbits for the case $[\nu] - 1/3 = 0.1$. Particle starting from outside of the traingular-shaped area quickly leave the system.

Third-order resonance

An example of experimentally measured third-order resonance orbits at the IUCF cooler ring.



In collider rings, by contrast, beam-beam interactions tend to be a bigger concern than field errors or nonlinearities.

Resonance overlapping and dynamic aperture (Lecture 10)

January 28, 2016

Lecture outline

Linear Hamiltonian in accelerators is integrable, and the motion is regular. What is the effect of nonlinear terms in the Hamiltonian? They lead to many nonlinear resonances in the motion and may result in stochastic motion of the particles. We will consider a simple model, a so called *standard map*, which illustrates qualitative features of what can occur in a system with many resonances.

Standard map and resonance overlapping

We start from the following Hamiltonian

$$H(I,\theta,t) = \frac{1}{2}I^2 + K\tilde{\delta}(t)\cos\theta\,, \tag{10.1}$$

where K is a parameter, $\tilde{\delta}(t) = \sum_{n=-\infty}^{\infty}\delta(t+n)$ is the periodic δ function that describes unit kicks repeating with the unit period. Here I is the action variable, and θ as the angle. Compare with (9.21). Both I and θ are dimensionless. The equations of motion are

$$\dot{I} = -\frac{\partial H}{\partial\theta} = K\tilde{\delta}(t)\sin\theta\,, \qquad \dot{\theta} = \frac{\partial H}{\partial I} = I\,. \tag{10.2}$$

[need a plot] If I_n and θ_n are the values at $t = n-0$ (before the delta-function kick), then integrating the first of Eqs. (10.2) from $t = n-0$ to $t = n+0$ (through the delta-function kick) gives $I_{n+1} = I_n + K\sin\theta_n$, which is conserved over the interval from $t = n+0$ to $t = (n+1)-0$ (where there are no kicks). Integrating the second equation in (10.2) from $t = n+0$ to $t = (n+1)-0$ and remembering that the action here is already equal to I_{n+1} gives $\theta_{n+1} = \theta_n + I_{n+1}$.

Standard map and resonance overlapping

Hence we arrive at the following transformation action-angle variables which links their values at time $t = n$ to the values at time $t = n + 1$:

$$
\begin{aligned}
I_{n+1} &= I_n + K \sin \theta_n \\
\theta_{n+1} &= \theta_n + I_{n+1} \,.
\end{aligned}
\tag{10.3}
$$

This transformation is called the *standard map* or *Chirikov map*[1].

[1] One can also find in the literature a definition of the standard map which differs from Eqs. (10.3) by numerical factors.

Standard map and resonance overlapping

The periodic delta-function in (10.1) can be expanded into the Fourier series

$$\tilde{\delta}(t) = 1 + 2\sum_{n=1}^{\infty} \cos\left(2\pi n t\right) . \tag{10.4}$$

Substituting this representation into the Hamiltonian Eq. (10.1) we can rewrite the latter in the following form

$$H = \frac{1}{2}I^2 + K\sum_{n=-\infty}^{\infty} \cos(\theta - 2\pi n t)\,, \tag{10.5}$$

(we used $2\cos(\theta)\cos(2\pi n t) = \cos(\theta - 2\pi n t) + \cos(\theta + 2\pi n t)$)). From this Hamiltonian we see that the system is a pendulum (the term $n = 0$ in the sum) driven by periodic perturbations with frequencies equal to $2\pi n$ (terms with $n \neq 0$).

Standard map and resonance overlapping

Selecting only one term in this sum would give us

$$\mathcal{H} = \frac{1}{2} I^2 + K \cos(\theta - 2\pi n t)\,. \tag{10.6}$$

We can make a canonical transformation $I, \theta \to J, \phi$ with

$$J = I - 2\pi n\,, \qquad \phi = \theta - 2\pi n t\,. \tag{10.7}$$

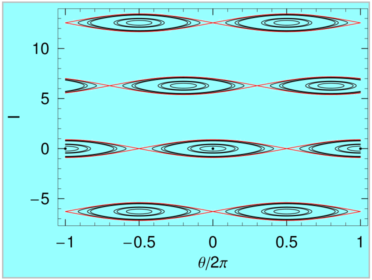
The new Hamiltonian for these variables is

$$\mathcal{H}' = \frac{1}{2} J^2 + K \cos\phi + \mathrm{const}\,. \tag{10.8}$$

One can see that Eq. (10.8) is the pendulum Hamiltonian with the phase space shown on the next slide. The width of the separatrix is equal $J = \pm 2\sqrt{K}$. In variable I, this phase space is shifted by $2\pi n$ units upward.

Standard map and resonance overlapping

Trying to understand what is the overall structure of the phase space of the original Hamiltonian, we can naively superimpose phase portraits for Hamiltonians (10.8) with various values of n. The width of each resonance is equal to $4\sqrt{K}$.



Of course, superimposing phase spaces is not a legitimate way of analysis (which is especially clear in the case when the separatrices of neighboring resonances overlap, see below).

Standard map and resonance overlapping

As long as the distance between the islands is much larger than the width of the separatrix, to a good approximation, resonances with different values of n can be considered separately. When the value of K increases the *resonances begin to overlap* and the dynamics becomes complicated.

Formally, overlapping occurs for $K > \pi^2/4$. Simulations show, when K increases, there is a gradual transition from a regular motion to a fully stochastic regime. Qualitatively, the transition occurs at

$$K \sim 1\,. \tag{10.9}$$

This is often called the Chirikov criterion of overlapping resonances.

Standard map and resonance overlapping

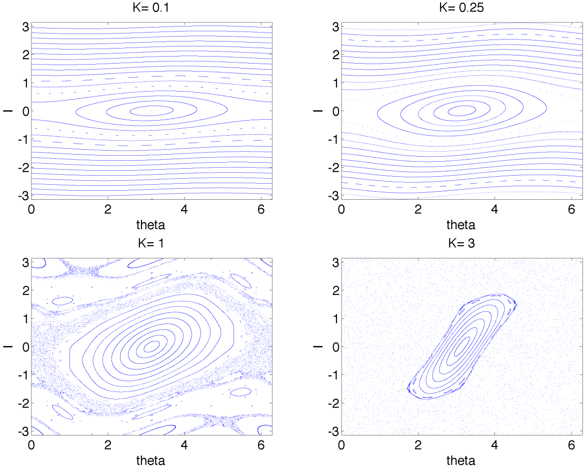


Figure : Computer simulations for the standard map for four different values of the parameter K, $K = 0.1, 0.25, 1, 3$ from left to right and from top to bottom.

Standard map and resonance overlapping

What happens in the regime of developed stochasticity, when $K \gg 1$?

After each kick the particle loses its memory about the previous phase, and the consecutive kicks can be considered as uncorrelated. The motion along the action axis I becomes random. We have from Eq. (10.3) the change of the action $\Delta I_n = K \sin \theta_n$. In the limit of large K the phase values become random and uncorrelated with each other. Taking square of ΔI_n and averaging over the random phase θ_n gives

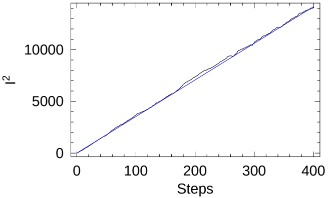
$$\langle \Delta I^2 \rangle = \frac{1}{2} K^2 \,. \tag{10.10}$$

If we plot the dependence I^2 versus the number of iterations N, we expect from Eq. (10.10)

$$I^2 \approx \frac{1}{2} K^2 N \,. \tag{10.11}$$

Standard map and resonance overlapping

A more detailed theory shows that Eq. (10.11) gives only the leading term for the diffusion process—there are notable corrections in this equation if K is not very large. In Figure below we confirm Eq. (10.11) by direct numerical simulation for $K = 8.41$ [Higher order corrections vanish for the value $K = 8.41$]. The straight line is the theoretical expectation given by Eq. (10.11).



Dynamic aperture in accelerators

A modern circular accelerators nonlinear components of the magnetic field of those magnets, as well as errors in manufacturing and installation of the magnets lead to many resonances in the machine. In a typical situation, the nonlinear fields make the phase space at some distance from the reference orbit more prone to stochastic motion, and result in the situation when only particles in a region near the reference orbit are properly confined. This region is called the *dynamic aperture* of the machine. It is computed with the help of accelerator codes by launching particles at various locations away from the reference orbit and tracking their motion. An example of calculation of the dynamic aperture for the light source SPEAR3 at SLAC is shown in Figure below.

Dynamic aperture in accelerators

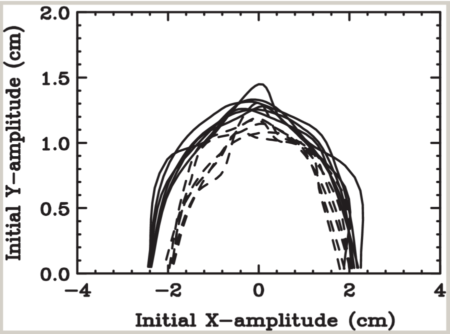


Figure : Dynamic aperture for the SPEAR3 light source at SLAC. Different curves correspond to 6 seeds of machine errors. The solid lines are for the nominal energy beam and the dashed ones are for the 3% energy deviation.

The kinetic equation
(Lecture 11)

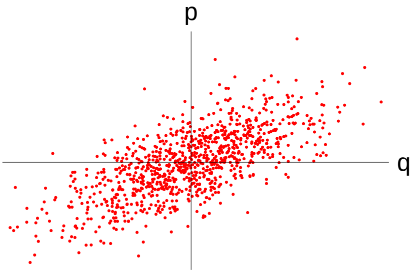
January 29, 2016

Lecture outline

In the preceding lectures we focused our attention on a single particle motion. In this lecture, we will introduce formalism for treating an ensemble of particles circulating in an accelerator ring.

Distribution function in phase space and kinetic equation

We start from considering a simple case of one degree of freedom with the canonically conjugate variables q and p.



A large ensemble of particles (think about a particle beam) with each particle having various values of q and p constitutes a "cloud" in the phase space.

Distribution function in phase space and kinetic equation

Let us consider an infinitesimally small region in phase space $dq \times dp$ and let the number of particles of the beam at time t in this. Mathematically infinitesimal phase element should be physically large enough to include many particles, $dN \gg 1$. We define the distribution function of the beam $f(q, p, t)$ such that

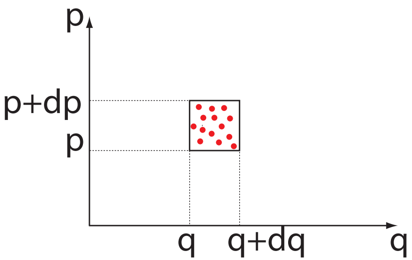
$$dN(t) = f(q, p, t) dp\, dq\,. \tag{11.1}$$

We can say that the distribution function gives the *density* of particles in the phase space.

Distribution function in phase space and kinetic equation

Particles travel from one place in the phase space to another, and the distribution function evolves with time. Our goal is to derive a *kinetic equation* that governs this evolution. In this derivation, we will assume that particles' motion is Hamiltonian.

Consider an infinitesimally small region of the phase space.



Distribution function in phase space and kinetic equation

The number of particles in this region at time t is given by Eq. (11.1). At time $t + dt$ this number will change because of the flow of particles through the boundaries. Due to the flow in the q-direction the number of particles that flow in through the left boundary is

$$f(q, p, t) \times dp\, \dot{q}(q, p, t) \times dt \tag{11.2}$$

and the number of particles that flow out through the right boundary is

$$f(q + dq, p, t) \times dp\, \dot{q}(q + dq, p, t) \times dt\,. \tag{11.3}$$

Distribution function in phase space and kinetic equation

Similarly, the number of particles which flow in through the lower horizontal boundary is

$$f(q, p, t) \times dq\, \dot{p}(q, p, t) \times dt \tag{11.4}$$

and the number of particles that flow out through the upper horizontal boundary is

$$f(q, p + dp, t) \times dq\, \dot{p}(q, p + dp, t) \times dt\,. \tag{11.5}$$

The number of particles in the volume $dq \times dp$ is now changed

$$\begin{aligned}
&dN(t + dt) - dN(t) \\
&= [f(q, p, t + dt) - f(q, p, t)]dp\, dq \\
&= f(q, p, t)dp\, \dot{q}(q, p, t)dt - f(q + dq, p, t)dp\, \dot{q}(q + dq, p, t)dt \\
&+ f(q, p, t)dq\, \dot{p}(q, p, t)dt - f(q, p + dp, t)dq\, \dot{p}(q, p + dp, t)dt\,.
\end{aligned} \tag{11.6}$$

Distribution function in phase space and kinetic equation

Dividing this equation by $dp\,dq\,dt$ and expanding in Taylor's series (keeping only linear terms in dp, dq, dt) gives the following equation

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial q}\left[\dot{q}(q,p,t)f\right] + \frac{\partial}{\partial p}\left[\dot{p}(q,p,t)f\right] = 0 \ . \tag{11.7}$$

What we derived is the *continuity* equation for the function f.

Incompressible Hamiltonian flow

Due to the Hamiltonian nature of the flow in the phase space a medium represented by a distribution function f in *incompressible*. This follows from the Liouville theorem. Indeed, according to this theorem the volume of a space phase element does not change in Hamiltonian motion. Since the value of f is the number of particles in this volume, and this number is conserved, f within a *moving* elementary volume is also conserved. The density at a given point of the phase space q, p however changes because other liquid elements arrive at this point at a later time.

Distribution function in phase space and kinetic equation

Mathematically, the fact of incompressibility is reflected in the following transformation of the continuity equation (11.7). Let us take into account the Hamiltonian equations for $\dot{q}$ and $\dot{p}$:

$$\frac{\partial}{\partial q}\dot{q}(q,p,t) = \frac{\partial}{\partial q}\frac{\partial H}{\partial p} = \frac{\partial}{\partial p}\frac{\partial H}{\partial q} = -\frac{\partial}{\partial p}\dot{p}(q,p,t)\,, \qquad (11.8)$$

which allows to rewrite Eq. (11.7) as follows

$$-\frac{\partial f}{\partial t} + \frac{\partial H}{\partial q}\frac{\partial f}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial f}{\partial q} = 0\,. \qquad (11.9)$$

Distribution function in phase space and kinetic equation

In accelerator physics this equation is often called the *Vlasov* equation. It is a partial differential equation which is not easy to solve in most of the cases. It is however extremely useful for studying many effects in accelerators that involve interaction between the particles of the beam.
Note, that using the formalism of Poisson brackets, we can also write the Vlasov equation as

$$\frac{\partial f}{\partial t} + \{f, H\} = 0\,. \tag{11.10}$$

See (3.31). This means that

$$\frac{df}{dt} = 0$$

Distribution function in phase space and kinetic equation

In case of n degrees of freedom, with the canonical variables q_i and p_i, $n = 1, 2, \ldots, n$, the distribution function f is defined as a density in $2n$-dimensional phase space and depends on all these variables, $f(q_1, \ldots, p_1, \ldots, t)$. The Vlasov equation takes the form

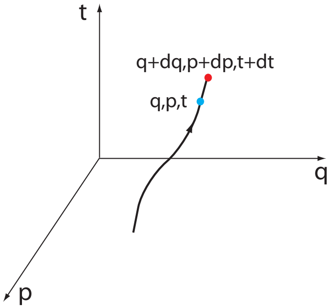
$$-\frac{\partial f}{\partial t} + \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \frac{\partial H}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial H}{\partial p_i} \right) = 0 \,. \tag{11.11}$$

Sometimes it is more convenient to normalize f by N, then the integral of f over the phase space is equal to one.

Integration of the kinetic equation along trajectories

We have stated above that the distribution function is constant within a moving infinitesimal element of phase space "liquid". We will now prove it.

Consider a trajectory in the phase space, and calculate the difference of f at two close points on this trajectory.



Integration of the kinetic equation along trajectories

We have

$$\begin{aligned} df &= f(q+dq, p+dp, t+dt) - f(q,p,t) \\ &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial q}dq + \frac{\partial f}{\partial p}dp\,. \end{aligned} \tag{11.12}$$

Remember that the two points are on the same trajectory, hence, $dq = \dot{q}dt = \partial H/\partial p\, dt$ and $dp = \dot{p}dt = -\partial H/\partial q\, dt$. We find

$$df = \frac{\partial f}{\partial t}dt - \frac{\partial H}{\partial q}\frac{\partial f}{\partial p}dt + \frac{\partial H}{\partial p}\frac{\partial f}{\partial q}dt = 0\,. \tag{11.13}$$

On the last step we invoked Eq. (11.9). We proved that the function f is constant along the trajectories.

Integration of the kinetic equation along trajectories

The above statement opens up a way to find solutions of the Vlasov equation if the phase space orbits are known. Let $q(q_0, p_0, t)$ and $p(q_0, p_0, t)$ be solutions of the Hamiltonian equations of motion with initial values q_0 and p_0 at $t = 0$, and $F(q_0, p_0)$ be the initial distribution function at $t = 0$. Then the solution of the Vlasov equation is given by the following equations

$$f(q, p, t) = F(q_0(q, p, t), p_0(q, p, t)) , \tag{11.14}$$

where the functions $q_0(q, p, t)$ and $p_0(q, p, t)$ are obtained as inverse functions from equations

$$q = q(q_0, p_0, t) , \qquad p = p(q_0, p_0, t) . \tag{11.15}$$

Steady state solutions of the kinetic equation

One of the powerful methods of solving the Vlasov equation is based on a judicious choice of canonical variables.
Let us use canonical variables J and ϕ. Then our kinetic equation is

$$\frac{\partial f}{\partial s} + \frac{\partial \hat{\mathcal{H}}}{\partial J}\frac{\partial f}{\partial \phi} - \frac{\partial \hat{\mathcal{H}}}{\partial \phi}\frac{\partial f}{\partial J} = \frac{\partial f}{\partial s} + \frac{\partial \hat{\mathcal{H}}}{\partial J}\frac{\partial f}{\partial \phi} = 0\,. \tag{11.16}$$

We see from this equation that any function f that depends only on J satisfies the equation $\partial f/\partial s = 0$, and hence is a steady state solution.

Steady state solutions of the kinetic equation

The particular dependence $f(J)$ is determined by various other processes in the ring. In many cases, a negative exponential dependence f versus J is a good approximation

$$f = \mathrm{const}\, e^{-J/\epsilon_0} = \mathrm{const}\, \exp\left(-\frac{1}{2\beta\epsilon_0}\left[x^2 + (\beta x' + \alpha x)^2\right]\right) . \tag{11.17}$$

The quantity ϵ_0 is called the beam *emittance*. It is an important characteristic of the beam quality.

Phase mixing and decoherence

Consider an ensemble of linear oscillators with the frequency ω, whose motion is described by the Hamiltonian

$$H(x,p) = \frac{p^2}{2} + \omega^2 \frac{x^2}{2}\,. \tag{11.18}$$

The distribution function $f(x,p,t)$ for these oscillators satisfy the Vlasov equation

$$\frac{\partial f}{\partial t} - \omega^2 x \frac{\partial f}{\partial p} + p \frac{\partial f}{\partial x} = 0\,. \tag{11.19}$$

We can easily solve this equation. The trajectory of an oscillator with the initial coordinate x_0 and momentum p_0 is

$$\begin{aligned} x &= x_0 \cos\omega t + \frac{p_0}{\omega}\sin\omega t \\ p &= -\omega x_0 \sin\omega t + p_0 \cos\omega t\,. \end{aligned} \tag{11.20}$$

Phase mixing and decoherence

Inverting these equations, we find

$$x_0 = x \cos \omega t - \frac{p}{\omega} \sin \omega t$$
$$p_0 = \omega x \sin \omega t + p \cos \omega t \,. \tag{11.21}$$

If $F(x, p)$ is the initial distribution function at $t = 0$, then, according to Eq. (11.14) we have

$$f(x, p, t) = F\left(x \cos \omega t - \frac{p}{\omega} \sin \omega t, \omega x \sin \omega t + p \cos \omega t\right) \,. \tag{11.22}$$

This solution describes rotation of the initial distribution function in the phase space. An initially *offset* distribution function results in *collective* oscillations of the ensemble with the betatron period. A *mismatched* distribution oscillates at half the betatron period.

Phase mixing and decoherence

A more interesting situation occurs if there is a frequency spread in the ensemble. Let us assume that each oscillator is characterized by some parameter δ (that does not change with time), and ω is a function of δ, $\omega(\delta)$.

$$H(x, p, \delta) = \frac{p^2}{2} + \omega(\delta)^2 \frac{x^2}{2} \,. \tag{11.23}$$

We then have to add δ to the list of the arguments of f and F, and Eq. (11.22) becomes

$$\begin{aligned} f(x, p, t, \delta) = F\bigg(& x \cos \omega(\delta) t - \frac{p}{\omega(\delta)} \sin \omega(\delta) t, \\ & \omega(\delta) x \sin \omega(\delta) t + p \cos \omega(\delta) t, \delta \bigg) \,. \end{aligned} \tag{11.24}$$

Phase mixing and decoherence

To find the distribution of oscillators over x and p only one has to integrate f over δ

$$\hat{f}(x,p,t) = \int_{-\infty}^{\infty} d\delta\, f(x,p,t,\delta)\,. \tag{11.25}$$

The behavior of the integrated function $\hat{f}$ is different from the case of constant ω at large times, even if the spread in frequencies $\Delta\omega$ is small. For $t \gtrsim 1/\Delta\omega$ the oscillators smear out over the phase. This effect is called the *phase mixing* and it results in *decoherence* of collective oscillations of the ensemble of oscillators.

Phase mixing and decoherence

In the limit $t \to \infty$ an initial distribution approaches a steady state which does not depend on time. We can find it using the action-angle variables.

$$H(J, \delta) = \omega(\delta) J \tag{11.26}$$

with

$$J = \frac{1}{2\omega} \left(p^2 + \omega(\delta)^2 x^2 \right) . \tag{11.27}$$

The Vlasov equation in $J - \phi$ coordinates is

$$\frac{\partial f}{\partial t} + \frac{\partial H}{\partial J} \frac{\partial f}{\partial \phi} = \frac{\partial f}{\partial t} + \omega(\delta) \frac{\partial f}{\partial \phi} = 0 . \tag{11.28}$$

In steady state $\partial f / \partial t = 0$, hence f only depends on J. This distribution should be $f_{\mathrm{eq}}(J) = (1/2\pi) \int d\phi \, \hat{f}(\phi, J, t = 0)$, which is an *orbit integral*.

Damped Systems

We begin considering damped systems by returning to the simple example of a simple harmonic oscillator with damping as in Eq. (2.3):

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = 0 \tag{11.29}$$

We keep as a Lagrangian

$$L(x, \dot{x}, t) = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega_0^2 x^2\,, \tag{11.30}$$

so that

$$\frac{\partial L}{\partial x} = -\omega_0^2 x\,, \qquad \frac{\partial L}{\partial \dot{x}} = \dot{x}\,. \tag{11.31}$$

Modified Hamiltonian equations for damping

Now we have to correct the previous equation for the Lagrangian to

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = -\ddot{x} - \omega_0^2 x = \gamma \dot{x}\,, \tag{11.32}$$

where the RHS used to be 0.
The Hamiltonian momentum is still $p = \partial L/\partial \dot{x} = \dot{x}$, and

$$H = p\dot{x} - L = \frac{1}{2}p^2 + \frac{1}{2}\omega_0^2 x^2 \tag{11.33}$$

The new equations of motion are

$$\frac{dx}{dt} = \frac{\partial H}{\partial p} = p\,, \tag{11.34}$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial x} - \gamma \dot{x} = -\omega_0^2 x - \gamma p \tag{11.35}$$

Modified Hamiltonian equations for damping

From this we can deduce that there is a new contribution to the evolution of the Hamiltonian,

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \frac{dp}{dt}\frac{\partial H}{\partial p} + \frac{dx}{dt}\frac{\partial H}{\partial x} = -\gamma p^2 \tag{11.36}$$

For $\gamma \ll \omega_0$, the average value of this change in Hamiltonian is approximately

$$\left\langle \frac{dH}{dt} \right\rangle = -\gamma \left\langle p^2 \right\rangle \simeq -\gamma H \tag{11.37}$$

Generalized damping

For more general analysis of damping, go straight to a modified Lagrangian equation, where

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \Gamma \tag{11.38}$$

The definition of conjugate momentum $p = \partial L / \partial \dot{q}$ is unchanged, and the Hamiltonian is still $H = p\dot{q} - L$, but the equations of motion are now

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial q} + \Gamma \tag{11.39}$$

Now, we see that

$$\frac{\partial \dot{q}}{\partial q} + \frac{\partial \dot{p}}{\partial p} = \frac{\partial \Gamma}{\partial p}, \tag{11.40}$$

rather than the RHS being 0 as before. This finally gives us

$$\frac{dH}{dt} = \Gamma \frac{\partial H}{\partial p} \tag{11.41}$$

Distribution function with damping

The calculation of the total derivative of a distribution function for a damped system works almost the same way as that for dH/dt, except that integrating by parts gives an even simpler result:

$$\frac{df}{dt} = -\frac{\partial \Gamma}{\partial p} f \, . \tag{11.42}$$

If $\Gamma = -\gamma p$, where γ is independent of p, then

$$\frac{df}{dt} = \gamma f \, . \tag{11.43}$$

In terms of either beam emittance or the action of an orbit,

$$\frac{d\epsilon}{dt} = \left\langle \frac{\partial \Gamma}{\partial p} \right\rangle \epsilon \simeq -\gamma \epsilon \, . \tag{11.44}$$

In order for the final approximation to be accurate, the damping should be weak compared to the oscillation frequency, or γ should be nearly uniform in the relevant region of phase space.

A brief look at diffusion

We can start with the usual definition of spatial diffusion and generalize it to phase space density:

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial x}\left(\mathcal{D}\frac{\partial f}{\partial x}\right) \simeq \mathcal{D}\frac{\partial^2 f}{\partial x^2} \tag{11.45}$$

As a result,

$$\frac{d\sigma_x^2}{dt} = 2\mathcal{D} \tag{11.46}$$

We can replace x with momentum p or energy E, depending on the source of diffusion. This is usually some kind of scattering process. For scattering that acts directly on the particle energy, the longitudinal emittance (which loosely scales as $\sigma_E\sigma_t$) will evolve according to

$$\frac{d\epsilon_z}{dt} = \frac{\epsilon_z}{2\sigma_E^2}\frac{d\sigma_E^2}{dt} = \frac{\mathcal{D}_E}{2\sigma_E^2}\epsilon_z \tag{11.47}$$

Primer in Special Relativity and Electromagnetic Equations (Lecture 13)

January 29, 2016

Lecture outline

We will review the relativistic transformation for time-space coordinates, frequency, and electromagnetic field.

Maxwell's equations

Classical electrodynamics in vacuum is governed by the Maxwell equations. In the SI system of units, the equations are

$$
\begin{aligned}
\nabla \cdot D &= \rho \\
\nabla \cdot B &= 0 \\
\nabla \times E &= -\frac{\partial B}{\partial t} \\
\nabla \times H &= j + \frac{\partial D}{\partial t}
\end{aligned}
\tag{13.1}
$$

where ρ is the charge density, j is the current density, with $D = \epsilon_0 E$, $H = B/\mu_0$. B is called the magnetic induction, and H is called the magnetic field.

The equations are linear: the sum of two solutions, E_1, B_1 and E_2, B_2, is also a solution corresponding to the sum of densities $\rho_1 + \rho_2$, $j_1 + j_2$.

Point charge

For a point charge moving along a trajectory $r = r_0(t)$,

$$\rho(r,t) = q\delta(r - r_0(t))\,, \qquad j(r,t) = qv(t)\delta(r - r_0(t))\,, \quad (13.2)$$

with $v(t) = dr_0(t)/dt$.

Proper boundary conditions should be specified in each particular case. On a surface of a good conducting metal the boundary condition requires that the tangential component of the electric field is equal to zero, $E_t|_S = 0$.

SI units

We will use the SI system of units (sometimes energy in eV).
To convert an equation written in SI variables to the corresponding equation in Gaussian variables, replace according to the following table (from Jackson's book):

Quantity	SI	Gaussian
Velocity of light	$(\mu_0\epsilon_0)^{-1/2}$	c
Electric field, potential	E, ϕ	$\frac{E}{\sqrt{4\pi\epsilon_0}}$, $\frac{\phi}{\sqrt{4\pi\epsilon_0}}$
Charge density, current	q, ρ, j	$q\sqrt{4\pi\epsilon_0}$, $\rho\sqrt{4\pi\epsilon_0}$, $j\sqrt{4\pi\epsilon_0}$
Magnetic induction	B	$B\sqrt{\frac{\mu_0}{4\pi}}$

We introduce the *vacuum impedance* Z_0,

$$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \approx 377 \text{ Ohm} . \qquad (13.3)$$

In CGS units $Z_0 = 4\pi/c$.

Wave equations

In free space with no charges and currents field components satisfy the wave equation

$$\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial z^2} = 0\,. \tag{13.4}$$

A particular solution of this equation is a sinusoidal wave characterized by frequency ω and wave number k and propagating in the direction of unit vector n:

$$f = A\sin(\omega t - kn\cdot r)\,, \tag{13.5}$$

where A is a constant and $\omega = ck$.

Vector and scalar potentials

It is often convenient to express the fields in terms of the *vector potential* A and the *scalar potential* ϕ:

$$E = -\nabla\phi - \frac{\partial A}{\partial t}$$
$$B = \nabla \times A \tag{13.6}$$

Substituting these equations into Maxwell's equations, we find that the second and the third equations are satisfied identically. We only need to take care of the first and the fourth equations.

Energy balance and the Poynting theorem

The electromagnetic field has an energy and momentum associated with it. The energy density of the field (energy per unit volume) is

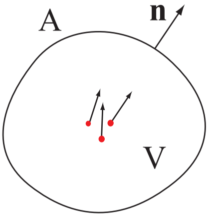
$$u = \frac{1}{2}(E \cdot D + H \cdot B) = \frac{\epsilon_0}{2}(E^2 + c^2 B^2)\,. \tag{13.7}$$

The Poynting vector

$$S = E \times H \tag{13.8}$$

gives the energy flow (energy per unit area per unit time) in the electromagnetic field.

Energy balance and the Poynting theorem



Consider charges that move inside a volume V enclosed by a surface A. The Poynting theorem states

$$\frac{\partial}{\partial t}\int_V u dV = -\int_V j \cdot E dV - \int_A n \cdot S dA \,, \tag{13.9}$$

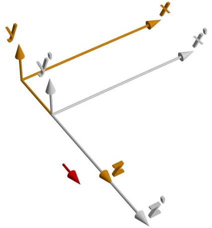
where n is the unit vector normal to the surface and directed outward.

The LHS of this equation is the rate of change of the electromagnetic energy. The first term on the right hand side is the work done by the electric field. The second term describes the electromagnetic energy flow from the volume through the enclosing surface.

Photons

The quantum view on the radiation is that the electromagnetic field is represented by photons. Each photon carries the energy $\hbar\omega$ and the momentum $\hbar k$, where the vector k is the wave number which points to the direction of propagation of the radiation, $\hbar = 1.05 \cdot 10^{-34}$ J$\cdot$sec is the Planck constant divided by 2π, and $k = \omega/c$.

Lorentz transformation and matrices



Consider two coordinate systems, K and K'. The system K' is moving with velocity v in the z direction relative to the system K. The coordinates of an event in both systems are related by the Lorentz transformation

$$\begin{aligned} x &= x', \qquad y = y', \\ z &= \gamma(z' + \beta ct'), \\ t &= \gamma(t' + \beta z'/c), \end{aligned} \tag{13.10}$$

where $\beta = v/c$, and $\gamma = 1/\sqrt{1-\beta^2}$.
The vector $(ct, r) = (ct, x, y, z)$ is called a *4-vector*, and the above transformation is valid for any 4-vector quantity.

Lorentz transformation and matrices

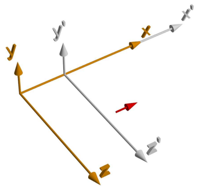
We will often deal with ultrarelativistic particles, which means that $\gamma \gg 1$. In this limit, a useful approximation is

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}} \approx 1 - \frac{1}{2\gamma^2} \,. \tag{13.11}$$

The Lorentz transformation (13.10) can also be written in the matrix notation

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & c\beta\gamma \\ 0 & 0 & \frac{\beta\gamma}{c} & \gamma \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} = L \begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} \,. \tag{13.12}$$

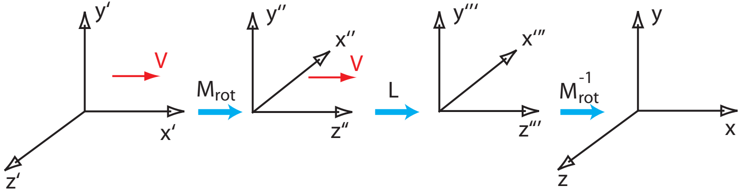
Lorentz transformation and matrices



The advantage of using matrices is that they can be consecutively applied in several steps. Here is an example: we want to generate a matrix which corresponds to a moving coordinate system along the x axis.

Lorentz transformation and matrices

Let us rotate K' system by 90 degrees around the y axis, in such a way that the new x axis is equal to the old z. The rotated frame is denoted by K'' and the



coordinate transformation from K' to K'' is given by $x'' = -z'$, $z'' = x'$, or in matrix notation

$$M_{\rm rot} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{13.13}$$

Lorentz transformation and matrices

A new frame K''' is moving along along the z'' axis and we then use the Lorentz transformation L to transform from K'' to K'''. Finally, we transform from K''' to the lab frame K using the matrix M_{rot}^{-1}. The sequence of these transformations is given by the product

$$(M_{\mathrm{rot}})^{-1} \cdot L \cdot M_{\mathrm{rot}} = \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta/c & 0 & 0 & \gamma \end{pmatrix} . \qquad (13.14)$$

This result, of course, can be easily obtained directly from the original transformation by exchanging $x \leftrightarrows z$.

Lorentz contraction and time dilation

Two events occurring in the moving frame at the same point and separated by the time interval $\Delta t'$ will be measured by the lab observes as separated by Δt,

$$\Delta t = \gamma \Delta t' . \tag{13.15}$$

This is the effect of relativistic *time dilation*.
An object of length l' aligned in the moving frame with the z' axis will have the length l in the lab frame:

$$l = \frac{l'}{\gamma} . \tag{13.16}$$

This is the effect of relativistic *contraction*. The length in the direction transverse to the motion is not changed.

Doppler effect

Consider a wave propagating in a moving frame K'. It has the time-space dependence:

$$\propto \cos(\omega' t' - k' r')\,, \tag{13.17}$$

where ω' is the frequency and k' is the wavenumber of the wave in the moving frame. What kind of time-space dependence an observer in the frame K would see? We need to make a Lorentz transformation of coordinates and time to get

$$\begin{aligned}
&\cos(\omega' t' - k' r') \\
&= \cos(\omega' \gamma(t - \beta z/c) - k_x{}' x - k_y{}' y - k_z{}' \gamma(z - \beta c t)) \\
&= \cos(\gamma(\omega' + k_z{}' \beta c) t - k_x{}' x - k_y{}' y - \gamma(k_z{}' + \omega' \beta/c) z)\,.
\end{aligned} \tag{13.18}$$

Doppler effect

We see that in the K frame this process is also a wave

$$\propto \cos(\omega t - kr)\,, \tag{13.19}$$

with the frequency and wavenumber

$$\begin{aligned}
k_x &= {k_x}'\,, \\
k_y &= {k_y}'\,, \\
k_z &= \gamma({k_z}' + \beta\omega'/c)\,, \\
\omega &= \gamma(\omega' + \beta c {k_z}')\,.
\end{aligned} \tag{13.20}$$

The object (ω, ck) is a 4-vector.

The result $\omega = \gamma(\omega' + {k_z}'\beta c)$ matches $E = \gamma(E' + {p_z}'\beta c)$, and similarly for k_z and p_z.

Doppler effect

The above transformation is valid for any type of waves (electromagnetic, acoustic, plasma waves, etc.) Now let us apply it to electromagnetic waves in vacuum. For those waves we know that

$$\omega = ck\,. \tag{13.21}$$

Assume that an electromagnetic wave propagates at angle θ' in the frame K'

$$\cos\theta' = \frac{k_z'}{k'}\,, \tag{13.22}$$

and has a frequency ω' in that frame. What is the angle θ and the frequency ω of this wave in the lab frame?

Doppler effect

We can always choose the coordinate system such that $k = (0, k_y, k_z)$, then

$$\tan\theta = \frac{k_y}{k_z} = \frac{k'_y}{\gamma(k_z{}' + \beta\omega'/c)} = \frac{\sin\theta'}{\gamma(\cos\theta' + \beta)}\,. \tag{13.23}$$

In the limit $\gamma \gg 1$ almost all angles θ' (except for those very close to π) are transformed to angles $\theta \sim 1/\gamma$. This explains why radiation of an ultrarelativistic beams goes mostly in the forward direction, within an angle of the order of $1/\gamma$.

Doppler effect

For the frequency, a convenient formula relates ω with ω' and θ (not θ'). To derive it, we use first the inverse Lorentz transformation

$$\omega' = \gamma(\omega - \beta c k_z) = \gamma(\omega - \beta c k \cos\theta)\,, \tag{13.24}$$

which gives

$$\omega = \frac{\omega'}{\gamma(1-\beta\cos\theta)}\,. \tag{13.25}$$

Using $\beta \approx 1 - 1/2\gamma^2$ and $\cos\theta = 1 - \theta^2/2$, we obtain

$$\omega = \frac{2\gamma\omega'}{1+\gamma^2\theta^2}\,. \tag{13.26}$$

The radiation in the forward direction ($\theta = 0$) gets a factor 2γ in the frequency.

Lorentz transformation of fields

The electromagnetic field is transformed from K' to K according to following equations

$$E_z = E'_z\,, \qquad E_\perp = \gamma\left(E'_\perp - v \times B'\right)\,,$$
$$B_z = B'_z\,, \qquad B_\perp = \gamma\left(B'_\perp + \frac{1}{c^2} v \times E'\right)\,, \tag{13.27}$$

where $E'_\perp$ and $B'_\perp$ are the components of the electric and magnetic fields perpendicular to the velocity v: $E'_\perp = (E_x, E_y)$, $B'_\perp = (B_x, B_y)$.

Lorentz transformation of fields

The electromagnetic potentials $(\phi/c, A)$ are transformed exactly as the 4-vector (ct, r):

$$\begin{aligned} A_x &= A'_x\,, \\ A_y &= A'_y\,, \\ A_z &= \gamma\left(A'_z + \frac{v}{c^2}\phi'\right)\,, \\ \phi &= \gamma(\phi' + vA'_z)\,, \end{aligned} \tag{13.28}$$

Lorentz transformation and photons

It is often convenient, even in classical electrodynamics, to consider electromagnetic radiation as a collection of photons. How do we transform photon properties from K' to K? The answer is simple: the wavevector k and the frequency of each photon ω is transformed as described above. The number of photons is a relativistic invariant—it is the same in all frames.

Selected electrostatic problems (Lecture 14)

February 1, 2016

Lecture outline

In this lecture, we demonstrate how to solve several electrostatic problems related to calculation of the fields generated by beams of charged particles in an accelerator.

Electric field of a 3D Gaussian distribution

A bunch of charged particles in accelerator physics is often represented as having a Gaussian distribution function in all three directions so that the charge density ρ is

$$\rho(x, y, z) = \frac{Q}{(2\pi)^{3/2}\sigma_x\sigma_y\sigma_z} e^{-x^2/2\sigma_x^2 - y^2/2\sigma_y^2 - z^2/2\sigma_z^2}, \tag{14.1}$$

where σ_x, σ_y, and σ_z are the rms bunch lengths in the corresponding directions. What is the electric field of such bunch? This is a purely electrostatic problem.

Due to the Lorentz transformations the bunch length in the beam frame is γ times longer than in the lab frame, $\sigma_{z,\mathrm{beam}} = \gamma\sigma_{z,\mathrm{lab}}$. We assume that this factor is already taken into account and σ_z in (14.1) is bunch length in the beam frame.

Electric field of a 3D Gaussian distribution

The electrostatic potential ϕ satisfies the Poisson equation

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0}\,, \tag{14.2}$$

whose solution can be written as

$$\phi(x,y,z) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(x',y',z')dx'dy'dz'}{[(x-x')^2+(y-y')^2+(z-z')^2]^{1/2}}\,. \tag{14.3}$$

It is not easy to carry out a three-dimensional integration in this equation. A trick that simplifies it and reduces to a one-dimensional integral is to use the following identity

$$\frac{1}{R} = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-\lambda^2R^2/2}d\lambda\,. \tag{14.4}$$

Electric field of a 3D Gaussian distribution

Assuming that $R = [(x - x')^2 + (y - y')^2 + (z - z')^2]^{1/2}$ and replacing $1/R$ in Eq. (14.3) with Eq. (14.4) we first arrive at the four-dimensional integral

$$\phi = \frac{1}{4\pi\epsilon_0}\sqrt{\frac{2}{\pi}}\int_0^\infty d\lambda \int e^{-\lambda^2[(x-x')^2+(y-y')^2+(z-z')^2]/2} \times \rho(x', y', z')dx'dy'dz' . \tag{14.5}$$

With the Gaussian distribution (14.1) the integration over x', y' and z' can now be easily carried out, e.g.,

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}\lambda^2(x-x')^2} e^{-\frac{x'^2}{2\sigma_x^2}}\, dx' = \frac{\sqrt{2\pi}}{\sqrt{\lambda^2 + \sigma_x^{-2}}} e^{-\frac{x^2\lambda^2}{2(\lambda^2\sigma_x^2+1)}} , \tag{14.6}$$

Electric field of a 3D Gaussian distribution

which gives for the potential

$$\phi = \frac{1}{4\pi\epsilon_0}\sqrt{\frac{2}{\pi}}\frac{Q}{\sigma_x\sigma_y\sigma_z}\int_0^\infty d\lambda \frac{e^{-\frac{x^2\lambda^2}{2(\lambda^2\sigma_x^2+1)}}e^{-\frac{y^2\lambda^2}{2(\lambda^2\sigma_y^2+1)}}e^{-\frac{z^2\lambda^2}{2(\lambda^2\sigma_z^2+1)}}}{\sqrt{\lambda^2+\sigma_x^{-2}}\sqrt{\lambda^2+\sigma_y^{-2}}\sqrt{\lambda^2+\sigma_z^{-2}}}. \tag{14.7}$$

This integral is much easier to evaluate numerically, and it is often used in numerical simulations of the field of charged bunches. There are various useful limiting cases of this expression, such as $\sigma_x = \sigma_y$ (axisymmetric beam) or $\sigma_x, \sigma_y \ll \sigma_z$ (a long, thin beam) that can be analyzed.

Electric field of a 3D Gaussian distribution

Having found the potential in the beam frame, it is now easy to transform it to the laboratory frame using the Lorentz transformation. First we have to recall that σ_z is the bunch length in the beam frame equal to $\gamma\sigma_{z,\mathrm{lab}}$. Second, from the Lorentz transformations we see that the potential in the lab frame is γ times larger than in the beam frame (note that $A'_z = 0$). Third, we need to transform the coordinates x, y, z to the lab frame. The x and y coordinates are not transformed however z should be replaced by $\gamma(z_{\mathrm{lab}} - vt_{\mathrm{lab}})$. The resulting expression is (we drop all "lab" subscripts in what follows)

$$\phi = \frac{1}{4\pi\epsilon_0}\sqrt{\frac{2}{\pi}}\frac{Q}{\sigma_x\sigma_y\sigma_z}\int_0^\infty \frac{d\lambda}{\sqrt{\lambda^2+\sigma_x^{-2}}\sqrt{\lambda^2+\sigma_y^{-2}}\sqrt{\lambda^2+\gamma^{-2}\sigma_z^{-2}}}$$

$$\times\, e^{-\frac{x^2\lambda^2}{2(\lambda^2\sigma_x^2+1)}}\, e^{-\frac{y^2\lambda^2}{2(\lambda^2\sigma_y^2+1)}}\, e^{-\frac{(z-vt)^2\lambda^2}{2(\lambda^2\sigma_z^2+\gamma^{-2})}}\,. \tag{14.8}$$

Electric field of a 3D Gaussian distribution

In addition to the electrostatic potential, in the lab frame there is also a vector potential A_z responsible for the magnetic field of the moving bunch. It is equal to $A_z = v\phi/c^2$ with ϕ given by (14.8).

Electric field of a continuous beam in a pipe

A continuous beam propagating inside a metallic pipe generates electric and magnetic fields. In many applications it is important to know these fields as a function of the beam position inside the pipe. We assume that the beam propagates parallel to the axis of a cylindrical pipe of a given cross section. The electrostatic potential ϕ is a function of the transverse coordinates x and y. We consider the beam as infinitely thin charged wire located at position $x = x_0$ and $y = y_0$. Then the problem of finding the electrostatic potential reduces to the solution of

$$\nabla^2 \phi = -\frac{\tilde{Q}}{\epsilon_0} \delta(x - x_0) \delta(y - y_0) \,, \tag{14.9}$$

where $\tilde{Q}$ is the charge per unit length of the beam. This equation is to be solved with the boundary condition $\phi = 0$ at the surface of the pipe.

Electric field of a continuous beam in a pipe

In cylindrical coordinate system r, θ

$$\Delta\phi(r,\theta) = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} \tag{14.10}$$

Electric field of a continuous beam in a pipe

In the simplest case of a beam located at the center of a round pipe of radius a ($x_0 = y_0 = 0$), the solution is easily found in cylindrical coordinates

$$\phi = -\frac{\tilde{Q}}{2\pi\epsilon_0} \ln\left(\frac{r}{a}\right) , \tag{14.11}$$

with the field $E_r = \tilde{Q}/2\pi\epsilon_0 r$.
What if the beam is not at the center of the round pipe? There is also an analytical solution in this case. A compact form of this solution is given as a real part of the complex function

$$\phi = -\frac{\tilde{Q}}{2\pi\epsilon_0} \mathrm{Re} \ln \frac{a(z - z_0)}{a^2 - z z_0} , \tag{14.12}$$

where $z = x + iy$ and $z_0 = x_0 + iy_0$.

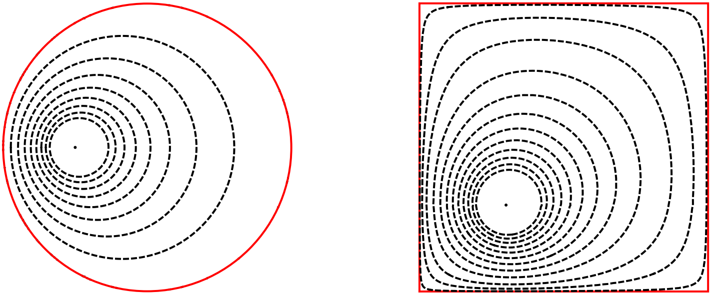
Electric field of a continuous beam in a pipe

If the beam is propagating in a pipe with a rectangular cross section $0 \le x \le a$, $0 \le y \le b$, the potential is given by the following expressions

$$\begin{aligned}
\phi_0 &= \frac{2\tilde{Q}}{\pi\epsilon_0}\sum_{k=1}^{\infty}\frac{1}{k\sinh\frac{k\pi b}{a}}\sinh\frac{k\pi(b-y_0)}{a}\sinh\frac{k\pi y}{a} \\
&\quad\times\sin\frac{k\pi x_0}{a}\sin\frac{k\pi x}{a},\ \text{for } y<y_0 \\
\phi_0 &= \frac{2\tilde{Q}}{\pi\epsilon_0}\sum_{k=1}^{\infty}\frac{1}{k\sinh\frac{k\pi b}{a}}\sinh\frac{k\pi(b-y)}{a}\sinh\frac{k\pi y_0}{a} \\
&\quad\times\sin\frac{k\pi x_0}{a}\sin\frac{k\pi x}{a},\ \text{for } y>y_0\,.
\end{aligned} \tag{14.13}$$

Electric field of a continuous beam in a pipe

Equipotential lines for a round and rectangular pipes.



Electric field of a continuous beam in a pipe

The approximation of an infinitely thin beam is useful for evaluation of the potential outside of the beam. However it cannot be used directly to calculate the potential between the center of the beam and the wall.

Let us consider the case of an axisymmetric Gaussian beam at the center of the round pipe with the charge density given by

$$\rho(r) = \frac{\tilde{Q}}{2\pi\sigma^2} e^{-r^2/2\sigma^2}\,, \tag{14.14}$$

with $\sigma \ll a$. In the infinitely thin beam approximation, the potential is given by Eq. (14.11). This expression is valid for $r \gg \sigma$; for $r = 0$ it gives an infinite value. To find the potential for a Gaussian beam we need to solve

$$\frac{1}{r}\frac{d}{dr} r \frac{d\phi}{dr} = -\frac{\tilde{Q}}{2\pi\sigma^2\epsilon_0} e^{-r^2/2\sigma^2}\,. \tag{14.15}$$

Electric field of a continuous beam in a pipe

The solution of this equation that has a finite electric field on the axis and satisfies the boundary condition at the wall is

$$\phi = \frac{\tilde{Q}}{2\pi\epsilon_0} \int_r^a \frac{dr'}{r'} \left(e^{-r'^2/2\sigma^2} - 1 \right) . \tag{14.16}$$

In the limit $r \gg \sigma$ we recover Eq. (14.11). The potential difference between the center of the beam and the wall is

$$\phi(r = 0) = \frac{\tilde{Q}}{2\pi\epsilon_0} \int_0^a \frac{dr'}{r'} \left(e^{-r'^2/2\sigma^2} - 1 \right) . \tag{14.17}$$

Electric field near metallic edges and protrusions

The electric field has a tendency to concentrate near sharp metallic edges and thin conducting protrusions. We illustrate this effect on several solvable problems of electrostatics.
The first problem is the field of a charged metallic ellipsoid.
Assume that the shape of the ellipsoid is given by the following equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\,, \tag{14.18}$$

where a, b and c are the half axes of the ellipsoid in the corresponding directions. The charge of the ellipsoid is equal to Q. This charge will be distributed on the surface of the solenoid so that the electrostatic potential ϕ is constant on the surface.

Electric field near metallic edges and protrusions

We will give here a solution of this problem without derivation. Let us assume that $a \geq b \geq c > 0$. We introduce a function $\lambda(x, y, z)$ that is defined as a *positive* solution of the following equation

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1\,. \tag{14.19}$$

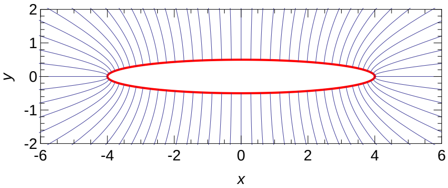
The value $\lambda = 0$ corresponds to the surface of the ellipsoid. Then the potential ϕ outside of the ellipsoid charged with the charge Q is given by the following integral

$$\phi(x, y, z) = \frac{Q}{2}\int_{\lambda(x,y,z)}^{\infty} \frac{d\xi}{\sqrt{(a^2+\xi)(b^2+\xi)(c^2+\xi)}}\,. \tag{14.20}$$

Electric field near metallic edges and protrusions

In the case of an elongated axisymmetric ellipsoid ($b = c$) the integration can be done in elementary functions with the result

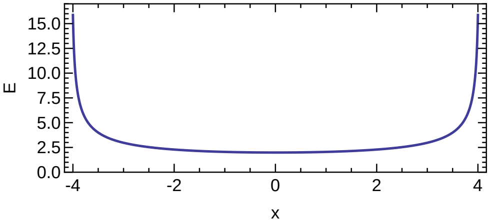
$$\phi(x, y, z) = \frac{Q}{\sqrt{a^2 - b^2}} \ln \frac{\sqrt{b^2 + \lambda}}{\sqrt{a^2 + \lambda} - \sqrt{a^2 - b^2}} . \qquad (14.21)$$



Field lines of an ellipsoid with $a = 4$ and $b = 0.5$ in the plane $z = 0$. One can see that the field is intensified near the ends of the ellipsoid.

Electric field near metallic edges and protrusions

Distribution of the electric field on the surface of the ellipsoid.



Electric field near metallic edges and protrusions

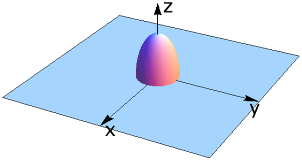


Figure : Ellipsoidal protrusion.

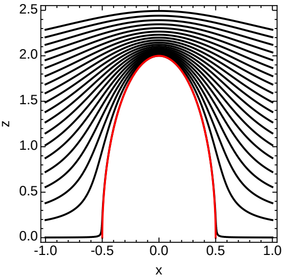


Figure : Contour plot of the potential around protruding ellipsoidal shape.

Electric field near metallic edges and protrusions

Amplification of the field near sharp edges is most clearly visible in 2D solutions of the Laplace equation near such edges. Consider the following problem: find the potential $\phi(r, \theta)$ in cylindrical coordinates that satisfies the equation $\nabla^2\phi = 0$ in the region $0 \leq \theta \leq \alpha$ with the boundary condition $\phi = 0$ at $\theta = 0$ and $\theta = \alpha$.

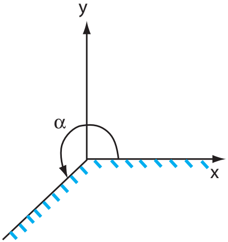


Figure : Coordinate system near an edge.

Electric field near metallic edges and protrusions

We have

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0\,. \tag{14.22}$$

It is easy to see that this equation is satisfied by $\phi = r^n \sin(n\theta)$ for arbitrary n. To satisfy the boundary condition, we require $\sin(n\alpha) = 0$ which gives $n = \pi/\alpha$. Hence

$$\phi = A r^{\pi/\alpha} \sin\left(\frac{\pi\theta}{\alpha}\right)\,, \tag{14.23}$$

where A is a constant. The electric field has a singularity if $n < 1$; it follows from the above expression that the field is singular when $\alpha > \pi$. In the limit $\alpha \to 2\pi$, which corresponds to the edge of a metallic plane, the potential scales as $\phi \propto \sqrt{r}$, and the field has a singularity $E \propto 1/\sqrt{r}$.

Electric field near metallic edges and protrusions

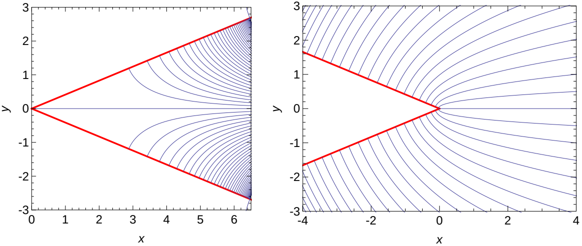


Figure : Field lines near the edge with $\alpha = \pi/4$ (left figure) and $\alpha = 7\pi/4$ (right figure).

It is interesting to note that the electric field near a sharp tip of a charged thin conical needle increases approximately as $1/r$, where r is the distance to the tip of the needle.

Self field of a relativistic beam (Lecture 15)

February 1, 2016

Lecture outline

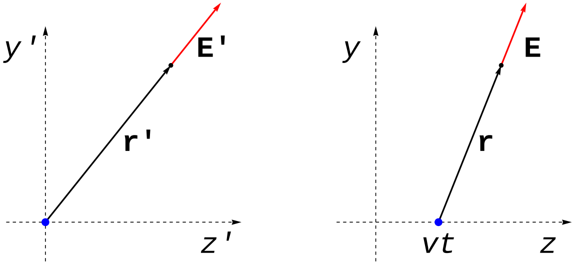
We will study the electromagnetic field of a bunch of charged particles moving with relativistic velocity along a straight line.

Relativistic field of a particle moving with constant velocity

Consider a point charge q moving with a constant velocity v along the z axis. We are interested in the case of a relativistic velocity, $v \approx c$, or $\gamma \gg 1$. In the particle's reference frame it has a static Coulomb field,

$$E' = \frac{1}{4\pi\epsilon_0}\frac{qr'}{r'^3}\,, \tag{15.1}$$

where the prime indicates the quantities in the reference frame where the particle is at rest.



Relativistic field of a particle moving with constant velocity

To find the electric and magnetic fields in the lab frame we will use the Lorentz transformation (13.10) for coordinates and time, and the transformation for the fields (13.27). We have $E_x = \gamma E'_x$, $E_y = \gamma E'_y$, and $E_z = E'_z$. We also need to make a Lorentz transform of the vector r' back into the lab frame. For the length of this vector we have

$$r' = \sqrt{x'^2 + y'^2 + z'^2} = \sqrt{x^2 + y^2 + \gamma^2(z - vt)^2}$$

Relativistic field of a particle moving with constant velocity

The Cartesian coordinates of E are

$$E_x = \frac{1}{4\pi\epsilon_0} \frac{q\gamma x}{(x^2 + y^2 + \gamma^2(z - vt)^2)^{3/2}}$$

$$E_y = \frac{1}{4\pi\epsilon_0} \frac{q\gamma y}{(x^2 + y^2 + \gamma^2(z - vt)^2)^{3/2}}$$

$$E_z = \frac{1}{4\pi\epsilon_0} \frac{q\gamma(z - vt)}{(x^2 + y^2 + \gamma^2(z - vt)^2)^{3/2}} \,. \tag{15.2}$$

These three equations can be combined into a vectorial one

$$E = \frac{1}{4\pi\epsilon_0} \frac{qr}{\gamma^2 \mathcal{R}^3} \,. \tag{15.3}$$

Here vector r is drawn from the current position of the particle to the observation point, $r = (x, y, z - vt)$, and $\mathcal{R}$ is given by

$$\mathcal{R} = \sqrt{(z - vt)^2 + (x^2 + y^2)/\gamma^2} \,. \tag{15.4}$$

Relativistic field of a particle moving with constant velocity

As follows from Eqs. (13.27), a moving charges carries magnetic field

$$B = \frac{1}{c^2} v \times E \, . \tag{15.5}$$

Relativistic field of a particle moving with constant velocity

The above fields can be also obtained by transforming the potentials. Indeed, in the particle's frame we have

$$\phi' = \frac{1}{4\pi\epsilon_0}\frac{q}{r'}\,, \qquad A' = 0\,. \tag{15.6}$$

Using the Lorentz transformation (13.28) we find

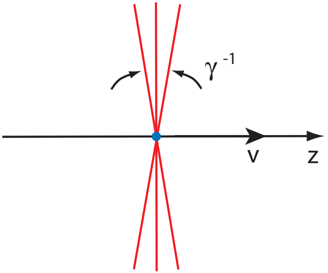
$$\phi = \gamma\phi'\,, \qquad A = \frac{1}{c}\boldsymbol{\beta}\phi\,. \tag{15.7}$$

Expressing r' in terms of the coordinates in the lab frame, $r' = \gamma\mathcal{R}$, gives

$$\phi = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathcal{R}}\,, \qquad A = \frac{Z_0}{4\pi}\boldsymbol{\beta}\frac{q}{\mathcal{R}}\,. \tag{15.8}$$

Relativistic field of a particle moving with constant velocity

The field of a relativistic point charge is illustrated below. Within a narrow cone with the angular width $\sim 1/\gamma$ the field is large, $E \sim q\gamma/r$. On the axis the field is weak, $E \sim 1/r\gamma^2$. The absolute value of the magnetic field is almost equal to that of the electric field.



Relativistic field of a particle moving with constant velocity

Sometimes we can neglect the small angular width of the electromagnetic field of a relativistic particle and consider it as an infinitely thin "pancake", $E \propto \delta(z - ct)$. This approximation formally corresponds to the limit $v \to c$. Because the field is directed along the vector drawn from the current position of the charge, more precisely, we can write $E = A\boldsymbol{\rho}\delta(z - ct)$ where $\boldsymbol{\rho} = \hat{x}x + \hat{y}y$ and A is a constant. The magnitude of A is determined by requiring that the areas under the curves $E_x(z)$ and $E_y(z)$ agree with the ones given by Eq. (15.3) in the limit $\gamma \to \infty$. This exercise yields (see homework problem 15.3)

$$E \simeq \frac{1}{4\pi\epsilon_0}\frac{2q\boldsymbol{\rho}}{\rho^2}\delta(z - ct)\,, \qquad B = \frac{1}{c}\hat{z} \times E\,. \qquad (15.9)$$

Interaction of Moving Charges in Free Space

Let us now consider a *source* particle of charge q moving with velocity v, and a *test* particle of *unit* charge moving behind the leading one on a parallel path at a distance l with an offset x. We want to find the force which the source particle exerts on the test one.

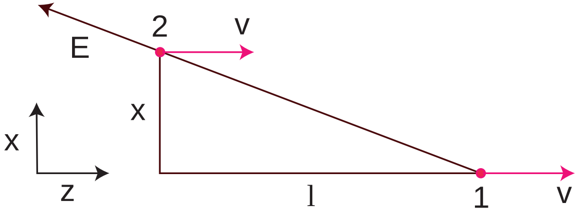


Figure : A leading particle 1 and a trailing particle 2 traveling in free space with parallel velocities v. Shown also is the coordinate system x, z.

Interaction of Moving Charges in Free Space

The longitudinal force is

$$F_l = E_z = -\frac{1}{4\pi\epsilon_0}\frac{ql}{\gamma^2(l^2 + x^2/\gamma^2)^{3/2}}\,, \tag{15.10}$$

and the transverse force is

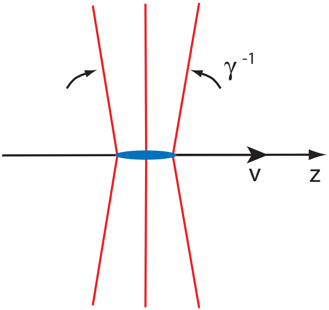
$$F_t = E_x - vB_y = \frac{1}{4\pi\epsilon_0}\frac{qx}{\gamma^4(l^2 + x^2/\gamma^2)^{3/2}}\,. \tag{15.11}$$

In accelerator physics, the force F is often called *the space charge force*.

The longitudinal force decreases with the growth of γ as γ^{-2} (for $l \gtrsim x/\gamma$). For the transverse force, if $l \gg x/\gamma$, $F_t \sim \gamma^{-4}$, and for $l = 0$, $F_t \sim \gamma^{-1}$. Hence, in the limit $\gamma \to \infty$, the electromagnetic interaction in free space between two particles on parallel paths vanishes.

Field of a long-thin relativistic bunch of particles

We consider a relativistic bunch of length σ_z much larger than the bunch radius $\sigma_z \gg \sigma_\perp$. The bunch is moving in the longitudinal direction along the z axis with a relativistic factor $\gamma \gg 1$. What is the electric field of this bunch?



Field of a long-thin relativistic bunch of particles

We first calculate the radial electric field outside of the bunch at distance ρ from the z axis. Assuming that $\rho \gg \sigma_\perp$ we can neglect the transverse size of the beam and represent it as a collection of point charges located on the z axis. Each such charge generates the electric field given by Eq. (15.3). From this equation we find that the radial component $d\mathcal{E}_\perp$ created by an infinitesimally small charge dq' located at coordinate z' is

$$d\mathcal{E}_\perp(z, z', \rho) = \frac{1}{4\pi\epsilon_0} \frac{\rho dq'}{\gamma^2((z - z')^2 + \rho^2/\gamma^2)^{3/2}}, \qquad (15.12)$$

where z and $\rho = \sqrt{x^2 + y^2}$ refer to the observation point. To find the field of the bunch we assume that the bunch 1D distribution function is given by $\lambda(z)$ ($\int \lambda(z) dz = 1$), so that the charge dq' within dz' is equal to $Q\lambda(z')dz'$, with Q the total charge of the bunch.

Field of a long-thin relativistic bunch of particles

For the field, we need to add contributions of all elementary charges in the bunch:

$$\begin{aligned} E_\perp(z,\rho) &= \int d\mathcal{E}_\perp(z,z',\rho) \\ &= \frac{Q\rho}{4\pi\epsilon_0\gamma^2}\int_{-\infty}^{\infty}\frac{\lambda(z')dz'}{((z-z')^2+\rho^2/\gamma^2)^{3/2}}\,. \end{aligned} \tag{15.13}$$

The function $((z-z')^2+\rho^2/\gamma^2)^{-3/2}$ in this integral has a sharp peak of width $\Delta z \sim \rho/\gamma$ at $z = z'$. At distances $\rho \ll \sigma_z\gamma$ from the bunch the width of the peak is smaller than the width of the distribution function σ_z, and we can replace it by the delta function. Precisely, we use

$$\frac{1}{((z-z')^2+\rho^2/\gamma^2)^{3/2}} \to \frac{2\gamma^2}{\rho^2}\delta(z-z')\,. \tag{15.14}$$

Field of a long-thin relativistic bunch of particles

The factor in front of the delta function on the right hand side follows from the requirements that the area under the functions on the left hand side and on the right hand side should be equal, and from the mathematical identity

$$\int_{-\infty}^{\infty} \frac{dz'}{((z-z')^2+a^2)^{3/2}} = \frac{2}{a^2}\,.$$

The approximation (15.14) is equivalent to using Eqs. (15.9) instead of (15.3). The result is

$$E_\perp(z,\rho) = \frac{1}{4\pi\epsilon_0}\frac{2Q\lambda(z)}{\rho}\,. \tag{15.15}$$

We see that the factor γ does not enter this formula—this agrees with our expectation because Eqs. (15.9) are valid in the limit $\gamma \to \infty$.

Field of a long-thin relativistic bunch of particles

In the opposite limit, $\rho \gg \sigma_z \gamma$, we can replace $\lambda(z)$ in Eq. (15.13) by the delta function $\delta(z)$, which gives the field of a point charge

$$E_\perp(z, \rho) = \frac{1}{4\pi\epsilon_0} \frac{Q\rho\gamma}{(z^2\gamma^2 + \rho^2)^{3/2}} \,. \tag{15.16}$$

In the intermediate region, $\rho \sim \sigma_z \gamma$, the result is shown on the next slide for a Gaussian distribution function $\lambda(z) = (1/\sqrt{2\pi}\sigma_z) e^{-z^2/2\sigma_z^2}$.

Field of a long-thin relativistic bunch of particles

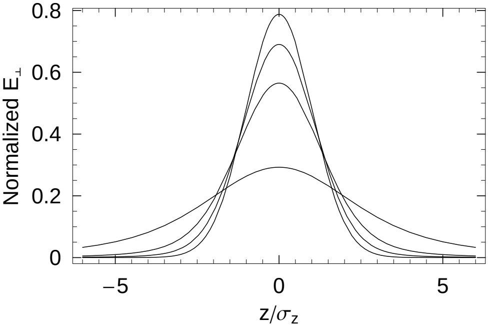


Figure : Transverse electric field of a relativistic bunch with Gaussian distribution for various values of the parameter $\rho/\sigma_z\gamma$. This parameter takes the values of 0.1, 0.5, 1 and 3 with larger values corresponding to broader curves. The field is normalized by $(4\pi\epsilon_0)^{-1}Q/\rho\sigma_z$.

Field of a long-thin relativistic bunch of particles

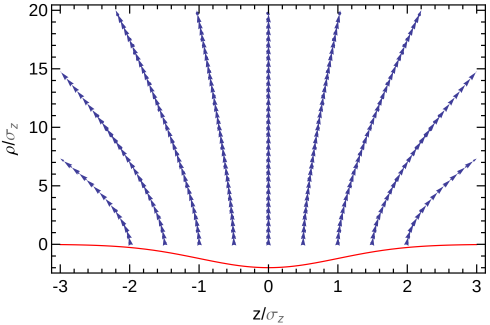


Figure : Electric field lines of a thin relativistic bunch with $\gamma = 10$. The red line at the bottom shows the longitudinal Gaussian charge distribution in the bunch.

Longitudinal field of a bunch

What is the longitudinal electric field inside the bunch? If we neglect the transverse size of the beam and assume the same infinitely-thin-beam approximation we used above, we can try to integrate the longitudinal field of a unit point charge

$$d\mathcal{E}_{\|}(z,z') = \frac{dq'}{4\pi\epsilon_0\gamma^2}\frac{z-z'}{|z-z'|^3}\,, \tag{15.17}$$

as we did above for the transverse field:

$$\begin{aligned} E_{\|}(z) &= \int d\mathcal{E}_{\|}(z,z') \\ &= \frac{Q}{4\pi\epsilon_0\gamma^2}\int dz'\lambda(z')\frac{z-z'}{|z-z'|^3}\,, \end{aligned} \tag{15.18}$$

but the integral diverges at $z' \to z$. This divergence indicates that one has to take into account the finite transverse size of the beam.

Longitudinal field of a bunch

Assume a uniform radial distribution of charge in the beam of radius a. Slice the beam into infinitesimal disks of thickness dz'.

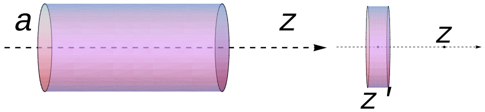
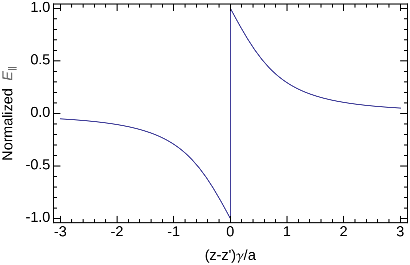


Figure : Left panel: a beam of cylindrical cross section a; right panel: a slice of the beam located at z'.

Longitudinal field of a bunch

If the slice has a unit charge and is located at coordinate z', the longitudinal electric field on the axis z at point z is

$$\mathcal{E}_{\parallel}(z, z') = -\frac{1}{4\pi\epsilon_0}\frac{2}{a^2}(z - z')\left(\frac{1}{\sqrt{a^2/\gamma^2 + (z - z')^2}} - \frac{1}{|z - z'|}\right).$$



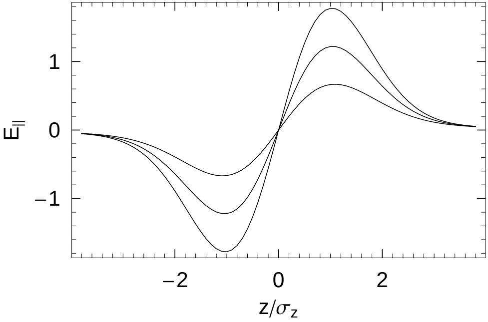
Longitudinal field of a bunch

The longitudinal electric field on the axis of the bunch is obtained by integration of contributions from the slices

$$
\begin{aligned}
E_{\|}(z) &= -\int_{-\infty}^{\infty} dz' Q\lambda(z')\mathcal{E}_{\|}(z,z') \qquad (15.19)\\
&= -\frac{Q}{4\pi\epsilon_0}\frac{2}{a^2}\int_{-\infty}^{\infty} dz'\lambda(z')(z-z') \\
&\left(\frac{1}{\sqrt{a^2/\gamma^2+(z-z')^2}}-\frac{1}{|z-z'|}\right).
\end{aligned}
$$

Numerical result for a Gaussian bunch

We assume Gaussian distribution $\lambda(z) = (1/\sqrt{2\pi}\sigma_z)e^{-z^2/2\sigma_z^2}$.



$E_\parallel$ for various values of the parameter $a/\gamma\sigma_z$: 0.1, 0.01, and 0.001. Smaller values corresponding to higher fields. The field is normalized by $(4\pi\epsilon_0)^{-1}2Q/\gamma^2\sigma_z^2$.

Field of a long-thin relativistic bunch of particles

One can show that in the limit $a/\gamma\sigma_z \ll 1$ a crude estimate for $E_\parallel$ is:

$$E_\parallel \sim \frac{1}{4\pi\epsilon_0}\frac{Q}{\sigma_z^2\gamma^2}\log\frac{\sigma_z\gamma}{a}\,. \tag{15.20}$$

Formally, this expression diverges in the limit of infinitely thin beam ($a \to 0$), but in reality the effect of the longitudinal electric field for relativistic beams is often small because of the factor γ^{-2} (a so called *space charge effect*).

The magnetic field of the beam

$$B = \frac{v}{c^2}\hat{z} \times E\,.$$

For an axisymmetric beam this means azimuthal magnetic field $B_\phi = \beta E_\perp/c$.

Effect of environment on electromagnetic field of a beam (Lecture 16)

February 1, 2016

Lecture outline

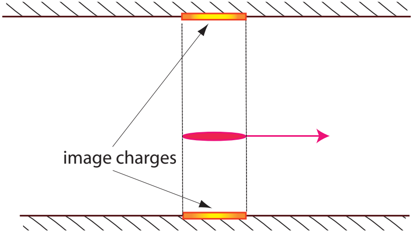
We first consider a relativistic beam moving in a perfectly conducting beam pipe. We the derive the Leontovich boundary condition, and derive the resistive wall wake field. We also discuss how a protrusion in a form of an iris affects the beam.

Introduction

We discussed in a previous lecture that electromagnetic interaction between particles of a relativistic beam moving in free space is suppressed. In practice, such interaction is often determined by the presence of material walls of the vacuum chamber and occurs if 1) the pipe is not cylindrical (which is usually due to the presence of RF cavities, flanges, bellows, beam position monitors, slots, etc., in the vacuum chamber), or 2) the wall of the chamber is not perfectly conducting.

Beam Moving in a Perfectly Conducting Pipe

Particle of a relativistic beam moving parallel to the axis in a perfectly conducting cylindrical pipe of arbitrary cross section, in the limit $v = c$, do not interact with each other.



Mathematically, the boundary condition for the fields on the surface of a perfectly conducting metal is

$$E_t = 0 \,. \tag{16.1}$$

Simplified Maxwell's equations

To understand interaction of a beam with a metallic wall, we need to consider effects of finite conductivity, or *resistive wall* effect.

We start with quick derivation of a so called *skin effect*.

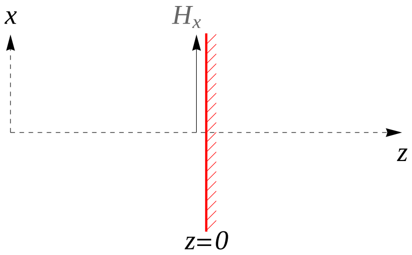
The skin effect deals with the penetration of the electromagnetic field inside a conducting medium characterized by a conductivity σ and magnetic permeability μ. We neglect the displacement current $\partial D/\partial t$ in Maxwell's equations in comparison with j:

$$\nabla \times H = j\,, \qquad \nabla \cdot B = 0\,, \qquad \nabla \times E + \frac{\partial B}{\partial t} = 0\,, \qquad j = \sigma E\,. \tag{16.2}$$

One finds the *diffusion* equation for the magnetic field B:

$$\frac{\partial B}{\partial t} = \sigma^{-1}\mu^{-1}\nabla^2 B\,. \tag{16.3}$$

Skin effect



A metal occupies a semi-infinite volume $z > 0$ with the vacuum at $z < 0$, and assume that at the metal surface the x-component of magnetic field is given by $H_x = H_0 e^{-i\omega t}$. Due to the continuity of the tangential components of H, H_x is the same on both sides of the metal boundary, that is at $z = +0$ and $z = -0$.

Skin effect

Seek solution inside the metal in the form $H_x = h(z)e^{-i\omega t}$. Equation (16.3) then reduces to

$$\frac{d^2 h}{dz^2} + i\mu\sigma\omega h = 0\,, \tag{16.4}$$

with the solution $h = H_0 e^{ikz}$ and

$$k = \sqrt{i\mu\sigma\omega} = (1+i)\sqrt{\frac{\mu\sigma\omega}{2}}\,. \tag{16.5}$$

Note that $\mathrm{Im}\, k > 0$ and the field exponentially decays into the metal. The quantity δ,

$$\delta = \sqrt{\frac{2}{\mu\sigma\omega}}\,, \tag{16.6}$$

is called the *skin depth*; it characterizes how deeply the electromagnetic field penetrates the metal.

Skin effect

In many cases the magnetic properties of the metal can be neglected, then $\mu = \mu_0$

$$\delta = \sqrt{\frac{2c}{Z_0 \sigma \omega}}\,. \tag{16.7}$$

The electric field inside the metal has only y component; it can be found from the first and the last of Eqs. (16.3)

$$E_y = \frac{j_y}{\sigma} = \frac{1}{\sigma}\frac{dH_x}{dz} = \frac{ik}{\sigma} H_x = \frac{i-1}{\sigma\delta} H_x\,. \tag{16.8}$$

The mechanism that prevents penetration of the magnetic field deep inside the metal is generation of a tangential electric field, that drives the current in the skin layer and shields the magnetic field.

The Leontovich boundary condition

The relation (16.8) can be rewritten in vectorial notation:

$$E_t = \zeta H \times n \,, \tag{16.9}$$

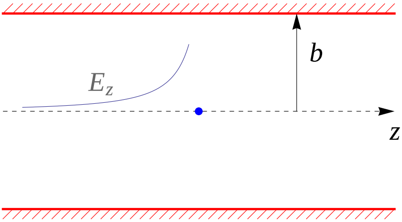
where n is the unit vector normal to the surface and directed toward the metal, and

$$\zeta(\omega) = \frac{1 - i}{\sigma\delta(\omega)} \,. \tag{16.10}$$

Eq. (16.9) is called the Leontovich boundary condition.
In the limit $\sigma \to \infty$ we have $\zeta \to 0$ and we recover the boundary condition (16.1) of zero tangential electric field on the surface of a perfect conductor. Remember that ζ is a function of ω — Fourier representation of the field.
At large frequencies the conductivity begin to depend on frequency, *ac conductivity*. Anomalous skin effect at low frequencies or very high frequencies.

Round pipe with resistive walls

Consider a round pipe of radius b, with wall conductivity σ. A point charge moves along the z axis of the pipe with $v = c$, $z = ct$. Because of the symmetry of the problem, the only non-zero component of the electromagnetic field on the axis is E_z. Our goal now is to find the field E_z as a function of z and t.



Round pipe with resistive walls

Main steps to find E_z:

- Take the magnetic field $B_\theta(\rho, z, t)$ of the charge (moving with $v = c$) in vacuum. It is equal to the field in *perfectly* conducting pipe. Assume, that it does not change much because the conductivity of the wall is large.
- Make Fourier transformation of $B_\theta(\rho, z, t)$ with respect to time, $B_\theta(\rho, z, t) \to \hat{B}_\theta(\rho, \omega)$.
- Use the Leontovich boundary conditions and find $\hat{E}_z|_{\rho=b}$ on the wall

$$\hat{E}_z|_{\rho=b} = -\zeta \frac{\hat{B}_\theta(b)}{\mu_0}\,.$$

- Use the wave equation for the field $E_z(\rho, z, t)$, make Fourier transformation of it, and take into account that $E_z = E_z(\rho, z - ct)$. Conclude from this equation that E_z does not depend on ρ: $\hat{E}_z|_{\rho=0} = \hat{E}_z|_{\rho=b}$.

Round pipe with resistive walls

We obtain

$$E_z(z,t) = \frac{qc}{4\pi^{3/2} b} \sqrt{\frac{Z_0}{\sigma s^3}} h(s)\,, \tag{16.11}$$

with $s = ct - z$.

For the points where $s < 0$, located in front of the charge, $E_z = 0$ in agreement with the causality principle. The positive sign of E_z indicates that a trailing charge (if it has the same sign as q) will be accelerated in the wake.

Short distance from the charge

The magnitude of E_z increases with decreasing s. At small distances the displacement current

$$j_z^{\mathrm{disp}} = \epsilon_0 \frac{\partial E_z}{\partial t} \,,$$

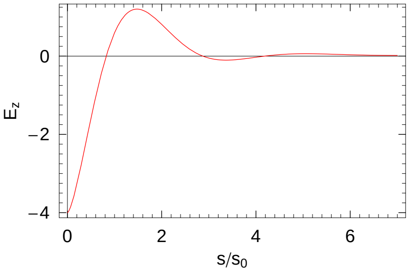
starts to change the magnetic field on the wall, and that in turn changes E_z. This happens at the distance

$$s \sim s_0 = \left(\frac{2b^2}{Z_0 \sigma} \right)^{1/3} \,. \tag{16.12}$$

For $b = 5$ cm

Metal	Copper	Aluminium	Stainless Steel
s_0, μm	60	70	240

Short distance from the charge



Longitudinal electric field as a function of distance s from the particle. The field is normalized by $q/4\pi\epsilon_0 b^2$, and the distance is normalized by s_0.

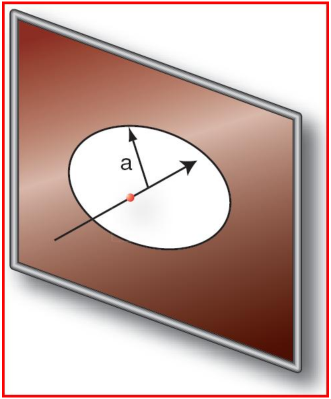
The value of the normalized field at the origin is equal to

$$-\frac{q}{\pi\epsilon_0 b^2}.$$

It does not depend on the conductivity!

Point charge passing through an iris

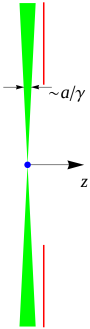
A relativistic point charge, $\gamma \gg 1$, moving in a pipe that has a diaphragm with round hole of radius a. We assume that a is much smaller than the pipe radius R and simplify to $R \to \infty$.



Point charge passing through an iris

The physics: the iris cuts off a part of the electromagnetic field, $r > a$, that hits the metal.

The frequency: the duration of the field pulse on the edge of the iris (where the field is strongest) is of the order of $\Delta t \sim a/c\gamma$. Hence characteristic frequencies are $\omega \sim c\gamma/a$.



The energy loss: calculate the energy U of the electromagnetic field that is "clipped away" by the iris. The field is given by Eq. (15.3) and (15.5),

$$E_\rho = cB_\theta = \frac{1}{4\pi\epsilon_0}\frac{\gamma q \rho}{(\rho^2 + \gamma^2 z^2)^{3/2}}\,.$$

The energy density w of the electromagnetic field is

$$w = \frac{\epsilon_0}{2}(E_\rho^2 + c^2 B_\theta^2)\,.$$

Point charge passing through an iris

Integrating w over the region $\rho > a$ and over z yields

$$U = \int_a^\infty 2\pi\,\rho\,d\rho \int_{-\infty}^\infty dz\,w = \frac{3}{64\epsilon_0}\frac{q^2\gamma}{a}\,. \tag{16.13}$$

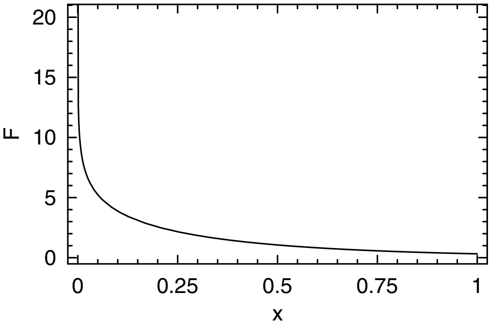
We expect that the radiated energy will be of the order of U, and the spectrum of radiation will involve the frequencies up to $\lambda\!\!\!^{-} \sim a/\gamma$ ($\lambda\!\!\!^{-} = 1/k$).

Point charge—analytical solution

The problem allows for an analytical solution. The radiated energy spectrum is

$$\frac{d\mathcal{W}}{d\omega} = \frac{1}{2\pi^2\epsilon_0}\frac{q^2}{c}F\left(\frac{ak}{\gamma}\right), \tag{16.14}$$

where $F(x) = x^2\left[K_0(x)\,K_2(x) - K_1(x)^2\right]$, with K_n the modified Bessel function of the second kind.



Point charge—analytical solution

The total radiated energy $\mathcal{W}$ is obtained by integrating $d\mathcal{W}/d\omega$ over the frequency:

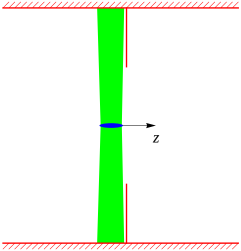
$$\int_0^\infty \frac{d\mathcal{W}}{d\omega} d\,\omega = \frac{3}{32\epsilon_0} \frac{q^2\gamma}{a}\,. \tag{16.15}$$

This is twice the clipped energy.

Explanation: The clipped field is reflected back by the screen, and is radiated in the backward direction. The same amount of energy is radiated by the screen in the forward direction when the particle rebuilds its original Coulomb field far from the screen.

Gaussian bunch passing through an iris

For a Gaussian bunch passing through the iris we take into account the pipe radius R. We assume no particles hit the iris, the bunch length satisfies the condition $\sigma_z > R/\gamma$, and use Eq. (15.15) for the beam transverse field. Also, $\sigma_\perp < a$.



The energy loss: The electromagnetic energy localized between the radii a and R is:

$$U = \frac{\epsilon_0}{2}\int_{-\infty}^{\infty} dz \int_a^R 2\pi\rho d\rho (E^2 + c^2 B^2)$$
$$= \epsilon_0 \int_{-\infty}^{\infty} dz \int_a^R 2\pi\rho d\rho E_\perp^2 \, .$$

$$U = \frac{1}{16\pi^2\epsilon_0}\frac{4Q^2\sqrt{\pi}}{\sigma_z}\ln\left(\frac{R}{a}\right) .$$

Gaussian bunch passing through an iris

The ratio of this energy to the kinetic energy of the beam $N\gamma mc^2$ (N is the number of particles in the beam) is

$$\begin{aligned} \frac{U}{N\gamma mc^2} &= \frac{1}{16\pi^2\epsilon_0}\frac{4Nq^2\sqrt{\pi}}{\gamma mc^2\sigma_z}\ln\left(\frac{R}{a}\right) \\ &= \frac{1}{\sqrt{\pi}}\frac{Nr_0}{\gamma\sigma_z}\ln\left(\frac{R}{a}\right), \end{aligned} \tag{16.16}$$

where q is the particle's charge ($Q = Nq$) and $r_0 = q^2/4\pi\epsilon_0 mc^2$ is the classical radius. For electrons $r_0 = 2.82 \cdot 10^{-13}$ cm and for protons $r_0 = 1.53 \cdot 10^{-16}$ cm.

Gaussian bunch passing through an iris

In contrast to a diaphragm, a smooth enough transition does not "scrape away" the electromagnetic field. When a beam passes through a smooth transition in a pipe its field is adiabatically adjusted to the shape of the local cross-section. It does not cause the energy loss but usually results in energy exchange between different parts of the beam (the head and the tail).

Even if the transition is not smooth, the radiation is suppressed for very long bunches, such that the characteristic frequency ω involved in the variation of the beam field, $\omega \sim c/\sigma_z$, is smaller than cut off frequency of the pipe.

Plane electromagnetic waves and Gaussian beams (Lecture 17)

February 2, 2016

Lecture outline

In this lecture we will study electromagnetic field propagating in space free of charges and currents. We focus on two types of solutions of Maxwell's equations: plane electromagnetic waves and Gaussian beams.

Plane electromagnetic waves

A plane electromagnetic wave propagates in free space (without charges and currents). All components of the field depend only on the variable $\xi = z - ct$,

$$E(r,t) = F(\xi)\,, \qquad B(r,t) = G(\xi)\,. \tag{17.1}$$

From the equation $\nabla \cdot E = 0$ if follows that $\partial F_z/\partial\xi = 0$, and hence $F_z = 0$. Similarly, $G_z = 0$ because of $\nabla \cdot B = 0$. We see that a plane wave is *transverse*.

We now apply Maxwell's equation $\partial B/\partial t = -\nabla \times E$ to the fields (17.1). We have

$$F_x' = cG_y'\,, \qquad F_y' = -cG_x'\,, \tag{17.2}$$

hence $F_x = cG_y$ and $F_y = -cG_x$.

Plane electromagnetic waves

In vector notation $F = -cn \times G$ or

$$E = -cn \times B\,, \tag{17.3}$$

where n is a unit vector in the direction of propagation (in our case along the z axis). Multiplying vectorially Eq. (17.3) by n, we also obtain

$$B = \frac{1}{c} n \times E\,. \tag{17.4}$$

If we use potentials ϕ and A to describe a plane wave, they would also depend on ξ only: $\phi = \phi(\xi)$, $A = A(\xi)$.

Plane electromagnetic waves

We have

$$\begin{aligned} B &= \nabla \times A \\ &= -\hat{x} A'_y + \hat{y} A'_x \\ &= n \times A' \\ &= -\frac{1}{c} n \times \frac{\partial A}{\partial t} \,. \end{aligned} \tag{17.5}$$

After the magnetic field is found, we can find the electric field using Eq. (17.3).

Plane electromagnetic waves

Often a plane wave has a sinusoidal time dependence with some frequency ω. In this case it is convenient to use the complex notation:

$$E = \mathrm{Re}\left(E_0 e^{-i\omega t + ikr + i\phi_0}\right), \qquad B = \mathrm{Re}\left(B_0 e^{-i\omega t + ikr + i\phi_0}\right),$$

where E_0 and B_0 are the amplitudes of the wave, and $k = n\omega/c$ is the wave number. The wave propagates in the direction of k; the amplitude of the electric and magnetic fields are $E_0 = cB_0$.

In general, E_0 and B_0 can be complex vectors orthogonal to k, e.g., $E_0 = E_0^{(r)} + iE_0^{(i)}$ with $E_0^{(r)}$ and $E_0^{(i)}$ real. Purely real of purely imaginary E_0 corresponds to a *linear polarization* of the wave; a complex vector E_0 describes an *elliptical polarization*.

Plane electromagnetic waves

The Poynting vector gives the energy flow in the wave

$$S = E \times H = \sqrt{\frac{\epsilon_0}{\mu_0}} E_0^2 n \cos^2(\omega t + kr + \phi_0)\,, \tag{17.6}$$

(in this formula E_0 is assumed real). The energy flows in the direction of the propagation k. Averaged over time energy flow, $\bar{S}$, is

$$\bar{S} = \frac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}} E_0^2 n = \frac{1}{2Z_0} E_0^2 n = \frac{c^2}{2Z_0} B_0^2 n\,. \tag{17.7}$$

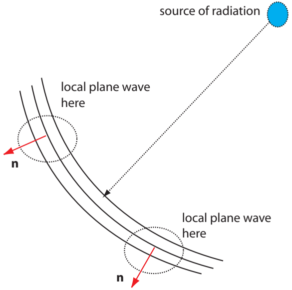
Energy density in the plane wave

$$\bar{u} = \frac{1}{2}\frac{\epsilon_0}{2}(E_0^2 + c^2 B_0^2) = \frac{\epsilon_0}{2} E_0^2$$

We have

$$\bar{S} = cu\,.$$

Plane electromagnetic waves



Electromagnetic field looks like a plane wave *locally* in some limited region.

An arbitrary solution of Maxwell's equations in free space (without charges) can be represented as a superposition of plane waves with various amplitudes and directions of propagation.

Gaussian beams

We consider another important example of electromagnetic field in vacuum—Gaussian beams. They are typically used for description of laser beams. They can be understood as a collection of plane waves with the same frequency propagating at small angles to a given direction—a so called *paraxial* approximation.

Gaussian beams

Start from the wave equation (13.4) for the x component of the electric field (a linear polarization of the laser light)

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2} = 0\,, \tag{17.8}$$

and assume

$$E_x(x, y, z, t) = u(x, y, z)e^{-i\omega t + ikz}\,, \tag{17.9}$$

where u is a slow function of its arguments, and $\omega = ck$. More specifically, we require

$$\left|\frac{1}{u}\frac{\partial u}{\partial z}\right| \ll k\ , \qquad \left|\frac{1}{u}\frac{\partial u}{\partial t}\right| \ll \omega\ . \tag{17.10}$$

Physical fields are the real part of E_x.

Gaussian beams

Main steps in the derivation of Gaussian beams:

- Neglect $\partial^2 u/\partial z^2$ in comparison with $k\partial u/\partial z$.
- Assume axisymmetry $u = u(\rho, z)$ with $\rho = \sqrt{x^2 + y^2}$.
- Seek solution in the form

$$u = A(z)e^{Q(z)\rho^2}$$

The result is

$$Q(z) = -\frac{1/w_0^2}{1 + 2iz/kw_0^2}\,, \qquad (17.11)$$

where w_0 is one constant of integration (called the *waist*) and

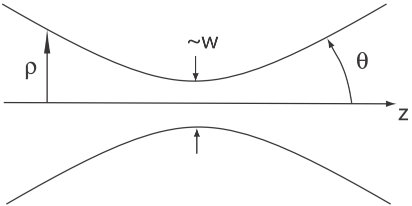
$$A(z) = \frac{E_0}{1 + 2iz/kw_0^2}\,, \qquad (17.12)$$

where E_0 is another constant of integration.

Gaussian beams

We now introduce important geometrical parameters: the *Rayleigh length* Z_R and the angle θ:

$$Z_R = \frac{kw_0^2}{2}, \qquad \theta = \frac{w_0}{Z_R} = \frac{2}{kw_0}. \tag{17.13}$$



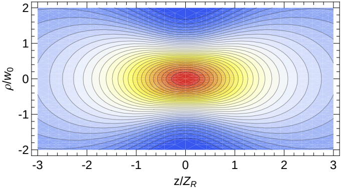
They can also be written as

$$Z_R = 2\frac{\lambda}{\theta^2}, \qquad w_0 = 2\frac{\lambda}{\theta}, \tag{17.14}$$

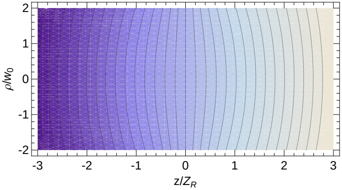
where $\lambda = k^{-1} = c/\omega$.

Gaussian beams

At $z = 0$ the radial dependence of u is $\propto e^{-\rho^2/w_0^2}$—$w_0$ gives the transverse size of the focal spot here. At $\rho = 0$ $u = E_0/(1 + iz/Z_R)$—Z_R is the characteristic length of the focal region along the z axis.



Contour lines of constant amplitude $|u|$.



Contour lines of constant phase ϕ, with $u = |u|e^{i\phi}$.

Gaussian beams

Condition for the validity of the paraxial approximation: $\partial^2 u/\partial z^2 \sim u/Z_R^2 \ll k\partial u/\partial z \sim ku/Z_R \to Z_R \gg \lambda\!\!\!^{-}$. From this condition it follows that $\lambda\!\!\!^{-} \ll w_0 \ll Z_R$ and $\theta \ll 1$. This means that the size of the focal spot w_0 is much larger than the reduced wavelength, and $\theta \ll 1$.

The magnetic field in a Gaussian beam the lowest approximation can be found, in the lowest order, by using Eq. (17.4), where n is directed along z,

$$B_y = \frac{1}{c} E_x \,. \tag{17.15}$$

RF cavities
(Lecture 25)

February 2, 2016

Lecture outline

A good conductor has a property to guide and trap electromagnetic field in a confined region. In this lecture we will consider an example of a radio frequency (RF) cavity, and discuss some of its properties from the point of view of acceleration of charged particles. We then discuss the electromagnetic pressure and derive Slater's formula.

Waveguide, TM modes

Let us consider a cylindrical waveguide of radius a made from a perfect conductor. Such a waveguide has a number of electromagnetic modes that can propagate in it. We will focus first our attention here on so called *TM modes* that have a nonzero longitudinal component of the electric field E_z, with $B_z = 0$. To find the distribution of the electric field in the waveguide for a mode that has frequency ω, we will assume that in cylindrical coordinates r, ϕ, z,

$$E_z(r, \phi, z, t) = \mathcal{E}(r) e^{-i\omega t - im\phi + i\varkappa z}, \tag{25.1}$$

use Eq. (13.4) for E_z

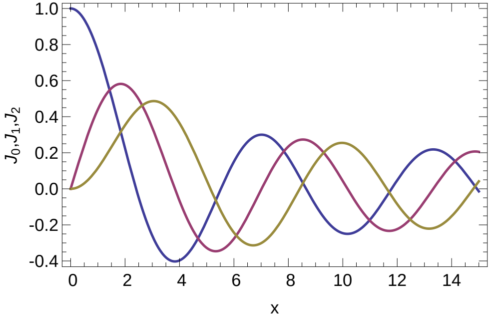
$$\frac{1}{r}\frac{d}{dr} r \frac{d\mathcal{E}}{dr} - \frac{m^2}{r^2}\mathcal{E} + \left(\frac{\omega^2}{c^2} - \varkappa^2\right)\mathcal{E} = 0\,. \tag{25.2}$$

Waveguide, TM modes

The solution of this equation is given by

$$\mathcal{E} = E_0 J_m \left(k_\perp r \right) , \tag{25.3}$$

where J_m is the Bessel function of m-th order and $k_\perp = c^{-1}\sqrt{\omega^2 - c^2\varkappa^2}$. The boundary condition $E_z = 0$ at $r = a$ requires that $k_\perp r$ be equal to a zero of J_m. For each function J_m there is an infinite sequence of such zeros, which we denote by $j_{m,n}$ with $n = 1, 2, \ldots$.



Waveguide, TM modes

Hence $k_\perp = j_{m,n}/a$ and recalling the definition of $k_\perp$ we find that

$$\varkappa_{m,n} = \pm\left(\frac{\omega^2}{c^2} - \frac{j_{m,n}^2}{a^2}\right)^{1/2} . \tag{25.4}$$

In order for a mode with indices m and n to have a real value of $\varkappa$, its frequency should be larger than the *cut-off frequency* $cj_{m,n}/a$. The plus sign defines the modes propagating in the positive direction, and the minus sign corresponds to the modes in the opposite direction. If $\omega < cj_{m,n}/a$, then we deal with *evanescent* modes that exponentially decay along the z-axis (and, correspondingly exponentially grow in the opposite direction).

Waveguide, TM modes

Given $E_z(r, \phi, z, t)$ as defined by (25.1) we can find all other components of the electric and magnetic fields using Maxwell's equations. They will all have the same dependence $e^{-i\omega t - im\phi + i\varkappa z}$ versus time, angle and z. The radial distribution of the four unknown components E_ϕ, E_r, B_ϕ and B_r (remember that $B_z = 0$) are found from the four algebraic equations, which are r and ϕ components of the two vectorial equations $\nabla \times E = i\omega B$ and $c^2 \nabla \times B = -i\omega E$.

Waveguide, TM modes

Here is the result

$$E_r = E_0 \frac{i\varkappa_{m,n} a}{j_{m,n}} J'_m \left(j_{m,n} \frac{r}{a} \right) e^{-i\omega t - im\phi + i\varkappa_{m,n} z} \tag{25.5}$$

$$E_\phi = -E_0 \frac{m\varkappa_{m,n} a^2}{r j_{m,n}^2} J_m \left(j_{m,n} \frac{r}{a} \right) e^{-i\omega t - im\phi + i\varkappa_{m,n} z} \tag{25.6}$$

$$B_r = E_0 \frac{m\omega a^2}{c^2 r j_{m,n}^2} J_m \left(j_{m,n} \frac{r}{a} \right) e^{-i\omega t - im\phi + i\varkappa_{m,n} z} \tag{25.7}$$

$$B_\phi = E_0 \frac{i\omega a}{c^2 j_{m,n}} J'_m \left(j_{m,n} \frac{r}{a} \right) e^{-i\omega t - im\phi + i\varkappa_{m,n} z}, \tag{25.8}$$

where J'_m is the derivative of the Bessel function of order m and we dropped the indices m, n on the left sides. These modes are designated TM_{mn} or E_{mn}. Example: TM_{01}.

Waveguide, TM modes

Note that in addition to vanishing E_z on the wall, which we have satisfied by choosing $k_\perp = j_{m,n}/a$, we should also require $E_\phi = 0$ on the surface of the wall (because it is tangential there). This however is automatically satisfied because the radial dependence of E_ϕ in (25.6) is the same as E_z in (25.3).

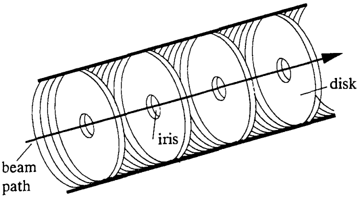
The physical meaning has the real parts of Eqs. (25.5). Since the longitudinal wavenumbers (25.4) do not depend on m, the modes with positive and negative values of m (assuming $m > 0$) are degenerate—they have the same values of $\varkappa_{m,n}$. A sum and difference of m and $-m$ modes, which convert $e^{im\phi}$ and $e^{-im\phi}$ into $\cos m\phi$ and $\sin m\phi$, are often used as another choice for the set of fundamental eigenmodes in circular waveguide.

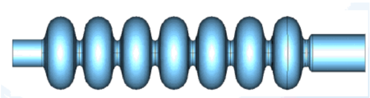
Waveguide, TE modes

Phase velocity of this modes is larger than the speed of light:

$$v_{ph} = \frac{\omega}{\varkappa} = c \frac{\omega}{\left(\omega^2 - c^2 j_{m,n}^2 / a^2\right)^{1/2}} > c \qquad (25.9)$$

This means that we cannot use EM field in waveguides for acceleration of charges. Disk loaded structures are used for that purpose.





Waveguide, TE modes

TE modes have nonzero longitudinal magnetic field B_z with $E_z = 0$. Their derivation follows closely that of TM modes. A simple observation of special symmetry of Maxwell's equations allows one to obtain the fields in TE modes without any calculation.
Indeed, assuming the time dependence $\propto e^{-i\omega t}$ for all fields, Maxwell's equations in free space are

$$\nabla \times E = i\omega B, \qquad c^2 \nabla \times B = -i\omega E, \qquad \nabla \cdot E = 0, \qquad \nabla \cdot B = 0. \tag{25.10}$$

Note that a transformation

$$(E, B) \to (cB, -E/c) \tag{25.11}$$

converts (25.10) into itself. This means that having found a solution of Maxwell's equation one can be obtain another solution by means of a simple transformation (25.11). The only problem with this approach is that one has to make sure that the boundary conditions are also satisfied.

RF modes in cylindrical resonator

Cylindrical resonator is a cylindrical pipe with the ends closed by metallic walls. Various modes of electromagnetic field that can exist in such a resonator are characterized by their frequency. The resonator modes can be easily obtained from the waveguide modes derived above.

In comparison with waveguides, a resonator requires one more boundary condition—vanishing tangential electric field on the end walls. Let's assume that the resonator left wall is located at $z = 0$, and the right wall is located at $z = L$. Start with TM modes. To satisfy the boundary condition $E_r = E_\phi = 0$ at $z = 0$ we choose two TM modes with the same frequency and the same m and n indices but opposite values of $\varkappa_{m,n}$ (that is two identical waves propagating in the opposite directions) add them and divide the result by 2.

RF modes in cylindrical resonator

Using

$$\frac{1}{2}(e^{i\varkappa_{m,n}z} + e^{-i\varkappa_{m,n}z}) = \cos(\varkappa_{m,n}z), \tag{25.12}$$

$$\frac{1}{2}(\varkappa_{m,n}e^{i\varkappa_{m,n}z} - \varkappa_{m,n}e^{-i\varkappa_{m,n}z}) = i\varkappa_{m,n}\sin(\varkappa_{m,n}z),$$

it is easy to see that both E_r and $E_\phi = 0$ acquire the factor $\sin(\varkappa_{m,n}z)$ and hence satisfy the boundary condition at $z = 0$. In order to satisfy the boundary condition at the opposite wall, at $z = L$, we require $\varkappa_{m,n}L = l\pi$, where $l = 1, 2, \ldots$ is an integer number.

RF modes in cylindrical resonator

The result is

$$E_z = E_0 J_m\left(j_{m,n}\frac{r}{a}\right)\cos\left(\frac{l\pi z}{L}\right)e^{-i\omega t - im\phi} \tag{25.13}$$

$$E_r = -E_0\frac{\varkappa_{m,n}a}{j_{m,n}}J_m'\left(j_{m,n}\frac{r}{a}\right)\sin\left(\frac{l\pi z}{L}\right)e^{-i\omega t - im\phi}$$

$$E_\phi = -E_0\frac{im\varkappa_{m,n}a^2}{rj_{m,n}^2}J_m\left(j_{m,n}\frac{r}{a}\right)\sin\left(\frac{l\pi z}{L}\right)e^{-i\omega t - im\phi}$$

$$B_r = E_0\frac{m\omega a^2}{c^2 rj_{m,n}^2}J_m\left(j_{m,n}\frac{r}{a}\right)\cos\left(\frac{l\pi z}{L}\right)e^{-i\omega t - im\phi}$$

$$B_\phi = E_0\frac{i\omega a}{c^2 j_{m,n}}J_m'\left(j_{m,n}\frac{r}{a}\right)\cos\left(\frac{l\pi z}{L}\right)e^{-i\omega t - im\phi}.$$

RF modes in cylindrical resonator

Eq. (25.4) should now be interpreted differently: we replace $\varkappa_{m,n}$ by $l\pi/L$, square it, and find the frequency ω of the mode

$$\frac{\omega^2}{c^2} = \left(\frac{l\pi}{L}\right)^2 + \frac{j_{m,n}^2}{a^2}\,. \tag{25.14}$$

The modes given by (25.13) and (25.14) are called TM_{mnl} modes.

RF modes in cylindrical resonator

A similar procedure can be done with the TE modes, but instead of adding, we need to subtract the mode with negative $\varkappa_{m,n}$ from the mode with the positive $\varkappa_{m,n}$ and divide the result by $2i$.
An important quantity associated with the mode is the energy W of the electromagnetic field. This energy is given by the integral over the volume of the cavity of $(\epsilon_0/2)(E_z^2 + c^2 B_\theta^2)$, where one has to take the real parts of the fields before squaring them.

TM_{010} mode

For illustration, let us calculate the energy of TM_{010} mode.

$$\frac{\omega_{010}^2}{c^2} = \frac{j_{0,1}^2}{a^2}, \tag{25.15}$$

where $j_{0,1} = 2.4$.

$$E_z = E_0 J_0\left(j_{0,1}\frac{r}{a}\right)e^{-i\omega_{010}t} \tag{25.16}$$

$$B_\phi = E_0 \frac{i\omega_{010} a}{c^2 j_{0,1}} J_0'\left(j_{0,1}\frac{r}{a}\right)e^{-i\omega_{010}t}.$$

The calculation can be simplified if one notices that although E_z and B_θ depend on time, the energy W does not. Because there is a phase shift of $\pi/2$ between these fields, one can find a moment when $B_\theta = 0$, and then

$$W = \frac{\epsilon_0}{2}\int dV|E_z|^2 = \frac{\epsilon_0}{2}\pi E_0^2 a^2 L J_1^2(j_1), \tag{25.17}$$

where we used $\int_0^1 J_0^2(bx)x dx = \frac{1}{2}J_1^2(b)$.

TM_{010} mode

With account of the finite conductivity of the wall, one finds that an initially excited mode decays with time because its energy is absorbed in the walls. This damping results in the imaginary part γ in the mode frequency, $\omega = \omega' - i\gamma$, where ω' and γ are real and positive. The imaginary part of the frequency can be calculated with the help of the Leontovich boundary condition.

A related quantity is the *quality factor* Q of the cavity equal to

$$Q = \frac{\omega'}{2\gamma} . \tag{25.18}$$

The quality factor for the TM_{010} mode of the cylindrical cavity

$$Q = \frac{aL}{\delta(a+L)} , \tag{25.19}$$

where δ is the skin depth at the frequency of the cavity. A crude estimate of the quality factor is $Q \sim l/\delta$, where l is a characteristic size of the cavity (assuming that all dimensions of the cavity are of the same order). Typical copper cavities used in accelerators have $Q \sim 10^4$; superconducting cavities may have $Q \sim 10^9$.

Electromagnetic field pressure

Electromagnetic field terminated by a conducting wall exerts a force on this wall.

When electric field lines are terminated on metal surface

there are image charges with the surface density equal to $\epsilon_0 E_n$, (n is the normal to the surface of the metal).

Electric field pressure

Consider in more detail distribution of the electric field inside the metal. The metal occupies the region $z > 0$. The charge density inside the metal is $\rho(z)$, and the electric field is $E_z(z)$.

$$\frac{dE_z}{dz} = \frac{\rho(z)}{\epsilon_0} \,. \tag{25.20}$$

The force per unit area is given by the integral

$$f_z^{(E)} = \int_0^\infty dz \rho E_z = \epsilon_0 \int_0^\infty dz E_z \frac{dE_z}{dz} = -\frac{\epsilon_0}{2} E_n^2 \,. \tag{25.21}$$

The minus sign means that the electric field has a "negative pressure"—it pulls the surface toward the free space.

Numerical example: for $E = 35$ MV/m the pressure is about 0.5 N/cm^2.

Magnetic field pressure

Tangential magnetic field also exerts a force on the surface.
Assume that the magnetic field $B_y(z)$ is directed along y, and varies along z due to the current $j_x(z)$ flowing in the x direction.

$$\frac{dH_y}{dz} = -j_x \tag{25.22}$$

and the force per unit area is

$$f_z^{(M)} = \int_0^\infty dz j_x B_y \,. \tag{25.23}$$

We have

$$f_z^{(M)} = -\int_0^\infty dz B_y \frac{dH_y}{dz} = \frac{1}{2\mu_0} B_t^2 \,, \tag{25.24}$$

We see that $f_z^{(M)}$ is positive—it acts as a real pressure applied to the surface.

Side comment on magnetic field pressure

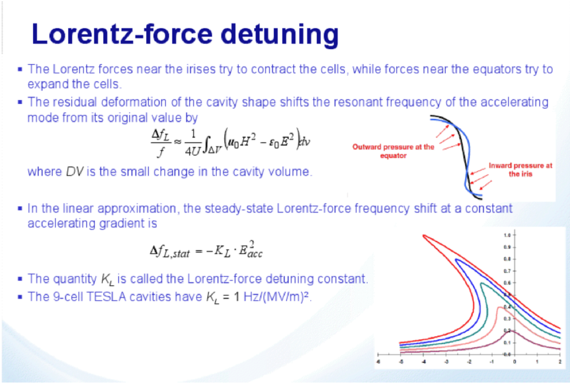
We briefly talked about a pressure tensor in force equations. For a uniform magnetic field in the $\hat{z}$ direction, the tensor describing the magnetic pressure away from any boundary is

$$\begin{pmatrix} B^2/2\mu_0 & 0 & 0 \\ 0 & B^2/2\mu_0 & 0 \\ 0 & 0 & -B^2/2\mu_0 \end{pmatrix} . \tag{25.25}$$

This has significant consequences for magnetized plasmas. The negative, longitudinal term is called *magnetic tension*.

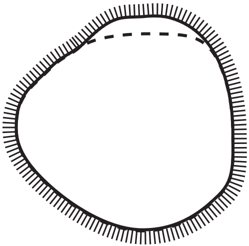
Electromagnetic field pressure

The effect of the electromagnetic pressure causes a so called *Lorentz detuning* in modern superconducting cavities which should be compensated by a special control system.



Slater's formula

What happens to the frequency of a cavity, if its shape is slightly distorted?



The frequency of modes changes. To calculate the frequency change we compute the work against the electromagnetic field needed to change the cavity shape.

Slater's formula

Assume that a mode is excited in the cavity and the distortion of the cavity shape occurs slowly in comparison with the frequency of the mode. This work, with a proper sign, is equal to the energy change δW of the mode. Since the distortion is small, we can take the unperturbed distribution of the electric and magnetic fields on the surface, compute $f_z^{(E)} + f_z^{(M)}$ and average over the period of oscillations. This averaging introduces a factor of $\frac{1}{2}$.

$$\delta W = \frac{1}{2} \int dS \; h \left(\frac{1}{2\mu_0} B_t^2 - \frac{\epsilon_0}{2} E_n^2 \right) , \tag{25.26}$$

where h is positive in the case when the volume of the cavity decreases, and it is negative in the opposite case, B_t and E_n are the amplitude values of the field on the surface.

Slater's formula

We know from quantum theory that the number of quanta does not change in adiabatically slow processes. Hence $W/\omega = \mathrm{const}$,

$$\frac{\delta\omega}{\omega} = \frac{\delta W}{W}\,. \qquad (25.27)$$

This gives us

$$\frac{\delta\omega}{\omega} = \frac{\epsilon_0}{4W}\int_{\Delta V} dV\left(c^2 B_t^2 - E_n^2\right)\,, \qquad (25.28)$$

where the integration in the numerator goes over the volume of the dent, and the integration in the denominator goes over the volume of the cavity. This is often called *Slater's formula*.

The perturbation must be small and cannot have any sharp features, otherwise one cannot use the unperturbed fields.

Radiation and retarded potentials (Lecture 18)

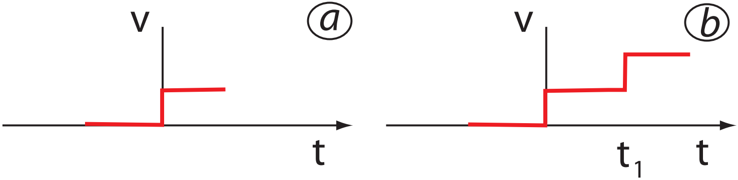
February 2, 2016

Lecture outline

In this lecture, based on simple intuitive arguments we derive the Liénard-Wiechert potentials that solve the problem of the electromagnetic field of a point charge moving in free space.

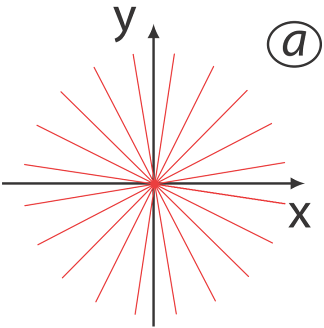
Radiation field

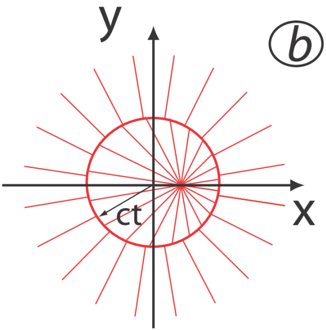
Assume that a point charge was at rest until $t = 0$, and then it is abruptly accelerated and moves with a constant velocity v at $t > 0$.



Radiation field

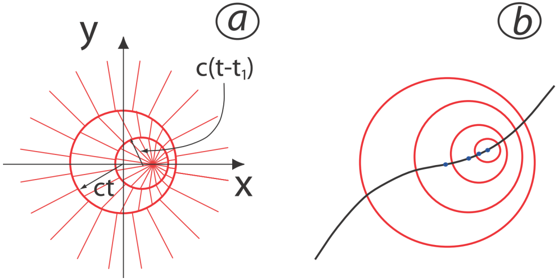
How do electric field lines look like before and after the acceleration?





Radiation field

If the charge was moved twice, then at time $t > t_1$ there will be two spheres, with radiation layers between them.



A constantly accelerating charge will be radiating the spheres at every moment of time, and those spheres will be expanding increasing their radii with the speed of light.

Retarded time and position

We need to figure out how to relate a point on such sphere to the time and position of the charge when this particular sphere was radiated. This time is called the *retarded* time and the position of the particle is the *retarded* position. If the particle's orbit is given by $r_0(t)$, and we make an observation at time t at point r in space, then the retarded time $t_{\rm ret}$ is determined from the equation

$$c(t - t_{\rm ret}) = |r - r_0(t_{\rm ret})| \tag{18.1}$$

and the retarded position is $r_0(t_{\rm ret})$. Note that both $t_{\rm ret}$ and $r_0(t_{\rm ret})$, for a given orbit of the particle (determined by the function $r_0(t)$) are functions of variables t and r.

Charge moving with constant velocity

In the limit when acceleration tends to zero, we obtain the limit of constant velocity. Let us find the retarded time in this case.
A point charge is moving with a constant velocity v along the z axis.

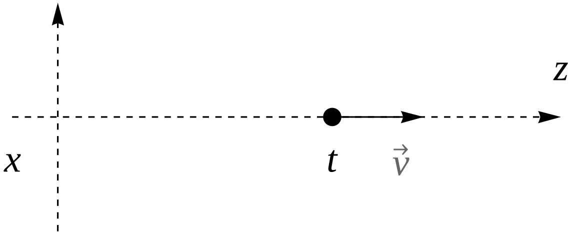


Figure : Point charge moving with constant velocity along the z-axis.

Charge moving with constant velocity

First we need to find t_{ret}. Using

$$r_0 = (0, 0, vt) \tag{18.2}$$

and introducing $t' = t - t_{\mathrm{ret}}$ ($t_{\mathrm{ret}} = t - t'$) we square Eq. (18.1)

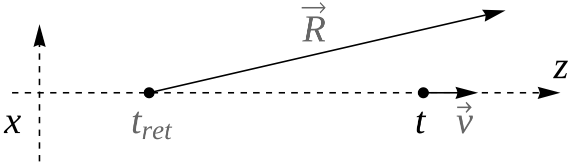
$$c^2 t'^2 = (z - v(t - t'))^2 + x^2 + y^2 \, . \tag{18.3}$$

This is a quadratic equation for t'. It has two solutions, one of them is an *advanced* solution with $t' < 0$, the other one is our retarded solution with $t' > 0$.

Charge moving with constant velocity

We will rewrite the equations for potentials (15.8) for a moving charge using t_{ret}.
Show first that the quantity $\mathcal{R}$ in Eq. (15.4) is equal to

$$\mathcal{R} = R - \boldsymbol{\beta} \cdot R = R(1 - \boldsymbol{\beta} \cdot n)\,. \tag{18.4}$$



See the proof in the Lecture notes.

Charge moving with constant velocity

The potentials (15.8) can now be written as

$$\phi = \frac{1}{4\pi\epsilon_0}\frac{q}{R(1-\boldsymbol{\beta}\cdot n)}\,, \qquad A = \frac{Z_0}{4\pi}\boldsymbol{\beta}\frac{q}{R(1-\boldsymbol{\beta}\cdot n)}\,. \tag{18.5}$$

Remember that R involves the retarded position of the particle. We can also formally consider $\boldsymbol{\beta}$ as taken at the retarded time, because it does not depend on time at all.

Liénard-Wiechert potentials

It turns out that in this new form the equations are valid for arbitrary motion of a point charge, even when the charge is being accelerated and we "accidentally" derived the *Liénard-Wiechert potentials* which describe electromagnetic field of an arbitrary moving particle:

$$\phi(r, t) = \frac{1}{4\pi\epsilon_0} \frac{q}{R(1 - \boldsymbol{\beta}_{\mathrm{ret}} \cdot n)} ,$$
$$A(r, t) = \frac{Z_0}{4\pi} \frac{q\boldsymbol{\beta}_{\mathrm{ret}}}{R(1 - \boldsymbol{\beta}_{\mathrm{ret}} \cdot n)} . \tag{18.6}$$

Here the particle's velocity $\boldsymbol{\beta}$ should be taken at the retarded time, $\boldsymbol{\beta}_{\mathrm{ret}} = \boldsymbol{\beta}(t_{\mathrm{ret}})$, and we remind that $R = r - r_0(t_{\mathrm{ret}})$ is a vector drawn from the retarded position of the particle to the observation point, and n is a unit vector in the direction of R.

Derivatives of the retarded time

Remember that $t_{\rm ret} = t_{\rm ret}(r, t)$. Let us calculate $\partial t_{\rm ret}/\partial t$. Square and differentiate (18.1):

$$\frac{\partial}{\partial t} c^2 (t - t_{\rm ret})^2 = \frac{\partial}{\partial t} (r - r_0(t_{\rm ret}))^2 \tag{18.7}$$

which gives

$$\begin{aligned} &-2c^2(t - t_{\rm ret}) \left(\frac{\partial t_{\rm ret}}{\partial t} - 1 \right) = -2(r - r_0(t_{\rm ret})) \frac{\partial}{\partial t} [r_0(t_{\rm ret})] \\ &= -2(r - r_0(t_{\rm ret})) \left. \frac{dr_0}{dt} \right|_{t=t_{\rm ret}} \frac{\partial t_{\rm ret}}{\partial t} \, . \end{aligned} \tag{18.8}$$

Using $\boldsymbol{\beta}_{\rm ret} = c^{-1} dr_0/dt|_{t=t_{\rm ret}}$ we find

$$\frac{\partial t_{\rm ret}}{\partial t} = \frac{1}{1 - \boldsymbol{\beta}_{\rm ret} \cdot n} \, . \tag{18.9}$$

Liénard-Wiechert potentials

Using the definitions of EM potentials,

$$
\begin{aligned}
E &= -\nabla\phi - \frac{\partial A}{\partial t} \\
B &= \nabla \times A \,,
\end{aligned}
\tag{18.10}
$$

we can obtain formulas for the fields:

$$
\begin{aligned}
E &= \frac{q}{4\pi\epsilon_0} \frac{n - \boldsymbol{\beta}_{\mathrm{ret}}}{\gamma^2 R^2 (1 - \boldsymbol{\beta}_{\mathrm{ret}} \cdot n)^3} + \frac{q}{4\pi\epsilon_0 c} \frac{n \times \{(n - \boldsymbol{\beta}_{\mathrm{ret}}) \times \dot{\boldsymbol{\beta}}_{\mathrm{ret}}\}}{R(1 - \boldsymbol{\beta}_{\mathrm{ret}} \cdot n)^3} \,, \\
B &= n \times E \,,
\end{aligned}
\tag{18.11}
$$

where $\dot{\boldsymbol{\beta}}_{\mathrm{ret}}$ is the acceleration (normalized by the speed of light) taken at the retarded time.

Retarded potentials for an ensemble of particles

The Liénard-Wiechert potentials are convenient for calculation of fields of a moving point charge. What if we are given a continuous time dependent current and charge distribution $\rho(r,t)$ and $j(r,t)$? Naively, one can think that to obtain the potential for a continuous distribution one has to replace the charge q by an infinitesimal charge $\rho(r',t)d^3r'$ in the elementary volume d^3r' and integrate over the space,

$$\frac{1}{4\pi\epsilon_0}\int\frac{\rho(r',t_{\mathrm{ret}})d^3r'}{|r-r'|(1-\boldsymbol{\beta}_{\mathrm{ret}}\cdot n)}\,, \tag{18.12}$$

where $n=(r-r')/|r-r'|$. This however, would be wrong. See Lecture notes.

Retarded potentials for an ensemble of particles

The correct expressions are:

$$\phi(r,t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(r',t_{\mathrm{ret}})}{|r-r'|}d^3r',$$
$$A(r',t) = \frac{Z_0}{4\pi}\int \frac{j(r',t_{\mathrm{ret}})}{|r-r'|}d^3r'. \qquad (18.13)$$

These integrals are called the *retarded* potentials. They give the radiation field in free space of a system of charges represented by continuous distribution of charge density ρ and current density j.

Scattering of electromagnetic waves (Lecture 19)

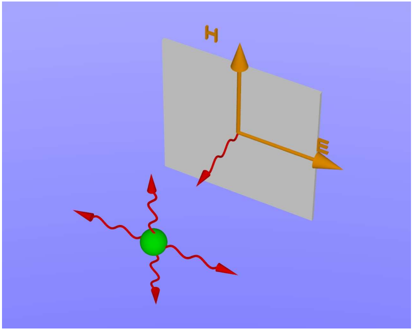
February 4, 2016

Lecture outline

We consider scattering of an electromagnetic wave on a free charged particle. Scattering involves a radiation reaction force, which keeps the energy balance in the process. The momentum balance is controlled by the light pressure effect. We briefly discuss the inverse Compton scattering on a point charge moving with a relativistic velocity.

Thomson scattering

An electron initially at rest is illuminated by a plane electromagnetic wave.



Electron motion

We will assume that this wave is weak, so that the electron velocity in the wave is nonrelativistic. The field in the wave is given by Eq. (17.6).

$$m\frac{dv}{dt} = qE_{\mathbf{0}}e^{-i\omega t+ik\cdot r}\,, \tag{19.1}$$

(we assumed the phase $\phi_0 = 0$). The magnetic force in this equation is dropped because $v \ll c$. Assume that $k \cdot r \ll 1$ on the right-hand side

$$m\frac{dv}{dt} = qE_{\mathbf{0}}e^{-i\omega t}\,. \tag{19.2}$$

Electron motion

Integration over time gives

$$v = \frac{iq}{m\omega} E_{\mathbf{0}} e^{-i\omega t}\,, \tag{19.3}$$

and

$$r = -\frac{q}{m\omega^2} E_{\mathbf{0}} e^{-i\omega t}\,. \tag{19.4}$$

The condition $v \ll c$ implies that

$$a \equiv \frac{qE_0}{m\omega c} \ll 1\,. \tag{19.5}$$

This condition means that, $kr \ll 1$, or

$$r \ll \lambda\,, \tag{19.6}$$

where $\lambda = c/\omega = \lambda/2\pi$.

Vector potential

We use Eq. (18.6) to calculate the vector potential.

First, we neglect the $\boldsymbol{\beta}_{\mathbf{ret}} \cdot n$ term in the denominator because $\beta \ll 1$. Second, we have approximately $R = r$. Finally, we neglect the term $r_0(t_{\mathrm{ret}})$ in Eq. (18.1):

$$t_{\mathrm{ret}} = t - \frac{r}{c} \,. \tag{19.7}$$

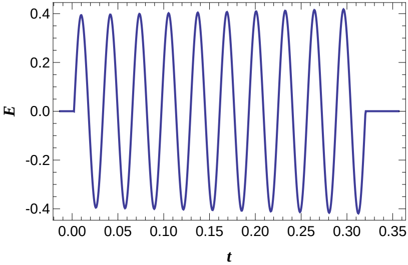
he result is

$$\begin{aligned} A(r,t) &= \frac{Z_0}{4\pi} \frac{q\boldsymbol{\beta}(t_{\mathrm{ret}})}{r} = \frac{Z_0}{4\pi} \frac{qv(t-\frac{r}{c})}{cr} \\ &= \frac{Z_0}{4\pi} \frac{iq^2}{m\omega cr} E_{\mathbf{0}} e^{-i\omega t + ikr} \,. \end{aligned} \tag{19.8}$$

We are dealing here with a spherical electromagnetic wave, whose amplitude decays with distance as $1/r$.

Vector potential

If the incident wave has a finite duration of N_w periods, the radiated wave will have the same number of periods, see (19.8).



This fact is useful in the study of undulator radiation.

Radiation intensity

We apply the local plane approximation and use (17.5) and (17.3). Eq. (17.5) gives

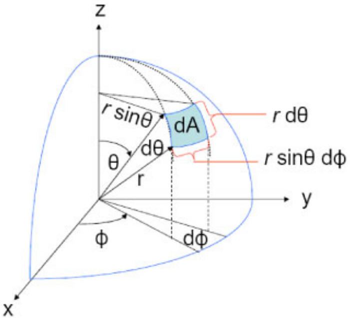
$$B = -\frac{1}{c} n \times \frac{\partial A}{\partial t} = \frac{Z_0}{4\pi} \frac{q^2}{mc^2 r} n \times E_{\mathbf{0}} e^{-i\omega t + ikr} \,. \tag{19.9}$$

Radiation occurs at the frequency of the incident wave. The energy flow in the radiation field is:

$$\begin{aligned} \bar{S} &= \frac{1}{2Z_0}|E|^2 = \frac{c^2}{2Z_0}|B|^2 = \frac{Z_0}{32\pi^2}\frac{q^4}{m^2c^2r^2}|n \times E_{\mathbf{0}}|^2 \\ &= \frac{Z_0}{32\pi^2}\frac{q^4 E_0^2}{m^2c^2r^2}\sin^2\psi \,. \end{aligned} \tag{19.10}$$

Angular distribution

The intensity of radiation: energy flow is $d\mathcal{P} = \bar{S}dA = \bar{S}r^2 d\Omega$

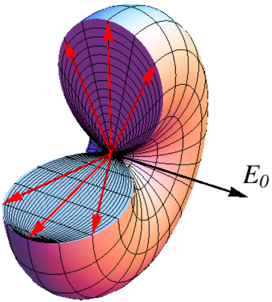


$d\Omega = r\sin\theta d\phi$
I use notation ψ instead of θ, with the axes z directed along E_0.

$$\frac{d\mathcal{P}}{d\Omega} = \bar{S}r^2 = \frac{Z_0}{32\pi^2}\frac{q^4E_0^2}{m^2c^2}\sin^2\psi\,. \qquad (19.11)$$

The angle ψ is measured relative to the direction of the electric field in the wave. The angular distribution is $\propto \sin^2\psi$.

Angular distribution



Angular distribution of the Thomson scattering.

Integrating this power over the solid angle gives ($\int d\phi \to 2\pi$)

$$\mathcal{P} = \int \frac{d\mathcal{P}}{d\Omega} d\Omega = \frac{Z_0}{32\pi^2} \frac{q^4 E_0^2}{m^2 c^2} \int_0^\pi \sin^2\psi \cdot 2\pi \sin\psi \, d\psi$$

$$= \frac{Z_0}{32\pi^2} \left(\frac{8\pi}{3}\right) \frac{q^4 E_0^2}{m^2 c^2} \,. \tag{19.12}$$

Comment

The radiation field decreases with radius as $1/r$. Only in this case does the total power of radiation remains constant as $r \to \infty$. Any EM field or component that decays faster than r^{-1} does not carry energy to infinity, and hence is not radiation.

Photon number

What is the number of photons emitted per unit time?

$$\begin{aligned}\dot{N}_p &= \frac{\mathcal{P}}{\hbar\omega} \\ &= \frac{Z_0}{12\pi}\frac{q^4 E_0^2}{m^2 c^2}\frac{1}{\hbar\omega} \\ &= \frac{1}{12\pi\epsilon_0}\frac{q^2 E_0^2}{m^2 c^2 \omega^2}\omega\frac{q^2}{\hbar c} \\ &= \frac{1}{3}a^2\omega\alpha\,, \end{aligned} \tag{19.13}$$

where a is defined by Eq. (19.5) and α is the fine structure constant

$$\alpha = \frac{1}{4\pi\epsilon_0}\frac{q^2}{\hbar c} \approx \frac{1}{137}\,. \tag{19.14}$$

This result is an approximation valid when $a \ll 1$, more generally it is $a^2/(1+a^2)$..

Thomson cross section

If we divide the radiated power by the average energy flow in the wave, we obtain a quantity that has dimension of length squared. This quantity can be interpreted as a scattering cross section, and is called the *Thomson cross section*

$$\sigma_{\mathrm{T}} = \frac{\mathcal{P}}{E_0^2/2Z_0} = \left(\frac{1}{4\pi\epsilon_0}\right)^2 \frac{8\pi q^4}{3m^2c^4} = \frac{8\pi}{3} r_0^2\,, \qquad (19.15)$$

where r_0 is the *classical radius*

$$r_0 = \frac{1}{4\pi\epsilon_0}\frac{q^2}{mc^2}\,. \qquad (19.16)$$

For electrons $r_0 = 2.8 \cdot 10^{-13}$ cm.

Angular distribution of radiation

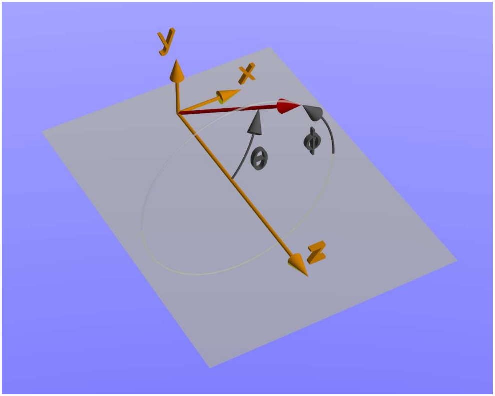


Figure : A spherical coordinate system. The wave propagates along the z axis, and the electric field in the wave is directed along the x axis.

Angular distribution of radiation

Here we calculate the intensity of radiation in a spherical coordinate system, where the wave propagates in the z direction and the electric field is directed along x. We introduce the polar angle θ measured relative to the z axis and the azimuthal angle ϕ measured in the $x - y$ plane.

$$n = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\,, \tag{19.17}$$

and with $E_0 = (E_0, 0, 0)$ we find

$$|n \times E_{\mathbf{0}}|^2 = E_0^2 (1 - \sin^2\theta\cos^2\phi)\,. \tag{19.18}$$

Eq. (19.11) takes the form

$$\frac{d\mathcal{P}}{d\Omega} = \frac{Z_0}{32\pi^2}\frac{q^4 E_0^2}{m^2 c^2}(1 - \sin^2\theta\cos^2\phi)\,. \tag{19.19}$$

Radiation reaction force

Because the charge is losing energy in the process of radiation, there should be a force acting against the velocity, a friction force. Such a force is called the *radiation reaction* force f_{rr}. We calculate it using the energy balance equation

$$-\langle f_{rr} \cdot v \rangle = \mathcal{P}\,, \tag{19.20}$$

where the angular brackets indicate averaging over time. Take Eq. (19.3) in real form

$$v = \frac{q}{m\omega} E_{\mathbf{0}} \sin \omega t\,, \tag{19.21}$$

and assume that f_{rr} is in the direction of the velocity and is in phase with it

$$f_{rr} = -Av\,. \tag{19.22}$$

Radiation reaction force

We then have

$$\langle f_{rr} \cdot v \rangle = -\frac{1}{2} A \left(\frac{qE_0}{m\omega} \right)^2 . \tag{19.23}$$

Equating this expression to $\mathcal{P}$ given by Eq. (19.12), we find

$$A = -\frac{Z_0}{6\pi} \frac{q^2 \omega^2}{c^2} , \tag{19.24}$$

and the force is

$$f_{rr} = -\frac{Z_0}{6\pi} \frac{q^2 \omega^2}{c^2} v = \frac{1}{6\pi\epsilon_0} \frac{q^2}{c^3} \ddot{v} = \frac{2}{3} \frac{r_0 m}{c} \ddot{v} . \tag{19.25}$$

The last expression, as it turns out, is more general than our derivation assumes—it is valid for arbitrary *nonrelativistic* motion of a point charge.

Light pressure

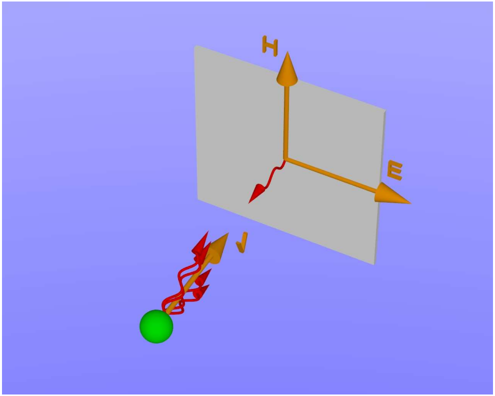
Let us take a quantum look at the Thomson scattering. The photons of the incident wave within the Thomson cross section are scattered off in different directions. The incident photons carry momentum in the direction of k, that is the direction of the wave propagation. The averaged momentum of scattered photons is zero. Hence the recoil momentum is transferred to the scatterer $\rightarrow$ there is a force in the direction of the wave propagation.
Let us use the quantum language. The ratio of the energy to momentum for each photon is c, hence $\mathcal{P}/c$ is the momentum scattered per unit time and is equal to the force f_{lp} in the longitudinal direction

$$f_{lp} = \frac{\mathcal{P}}{c} = \sigma_{\mathrm{T}} \frac{E_0^2}{2cZ_0} \,. \tag{19.26}$$

See the classical derivation in the Lecture notes.

Inverse Compton scattering

Assume that the scattering electron is moving with a relativistic velocity v ($\gamma \gg 1$) in the z direction and denote the incident wave frequency by ω_0. Make the Lorentz transformation from the lab frame to the beam frame.



Inverse Compton scattering

First we need to calculate the frequency ω' of the incident wave in the beam frame using the Lorentz transformation. We will use Eq. (13.25) in which $\theta = \pi$

$$\omega' = 2\gamma\omega_0 \,. \tag{19.27}$$

This is the frequency of the scattered radiation in the beam frame. To transform it to the lab frame we will assume small angles θ and use Eq. (13.26) again

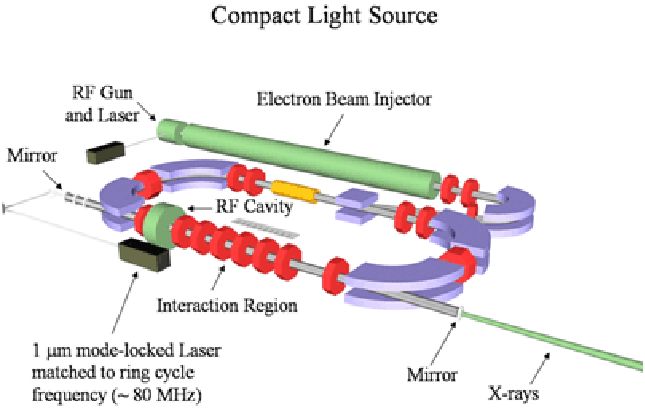
$$\omega = \frac{4\gamma^2\omega_0}{1+\gamma^2\theta^2} \,. \tag{19.28}$$

We see that there is a *spectrum* of frequencies with the maximum frequency equal to $4\gamma^2\omega_0$. Roughly, the frequencies of this order propagate within a small angle

$$\theta \sim \frac{1}{\gamma} \,. \tag{19.29}$$

Inverse Compton scattering

An example of using Compton scattering for generation x-rays. See `http://www.lynceantech.com/`



Synchrotron radiation (Lecture 20)

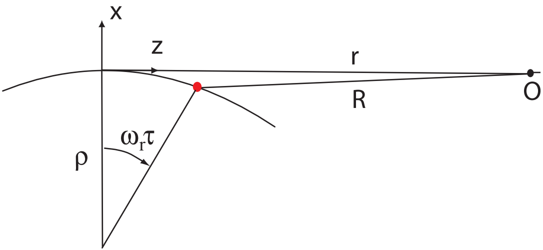
February 3, 2016

Lecture outline

We will consider a relativistic point charge ($\gamma \gg 1$) moving in a circular orbit of radius ρ. Our goal is to calculate the *synchrotron* radiation of this charge. Using the Liénard-Wiechert potentials we first find the fields at a large distance from the charge in the plane of the orbit. We then discuss properties of the synchrotron radiation using a more general result for the angular dependence of the spectral intensity of the radiation.

Synchrotron radiation pulses in the plane of the orbit

An observer is located in point O in the plane of the orbit in the far zone. The observer will see a periodic sequence of pulses of electromagnetic radiation with the period equal to the revolution period of the particle around the ring, $\omega_r = \beta c/\rho$ is the revolution frequency. Each pulse is emitted from the region $x \approx z \approx 0$.



Synchrotron radiation pulses in the plane of the orbit

Main steps in the derivation

- Use the plane wave approximation for the radiation field (and replace $\mathbf{n}$ with $\hat{z}$):

$$\mathbf{B} = -\frac{1}{c}\hat{\mathbf{z}} \times \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{E} = -c\hat{z} \times \mathbf{B}$$

- Denote the retarded time by τ, so that $R(\tau) = c(t - \tau)$, use $R(\tau) \approx r - \rho \sin \omega_r \tau$. At time $\tau = 0$ the particle is located at $x = z = 0$, $\rho \sin \omega_r \tau$ is approximately equal to the z coordinate at time τ, hence $R \approx r - z$.
- In the expression for $\mathbf{A}$, further approximate the factor R in the denominator by $R \simeq r$, yielding

$$\mathbf{A}(\mathbf{r}, t) = \frac{Z_0 q}{4\pi r} \frac{\boldsymbol{\beta}(t_{\mathrm{ret}})}{1 - \boldsymbol{\beta}(t_{\mathrm{ret}}) \cdot \mathbf{n}}$$

  Quantities depending on t_{ret} are not approximated in this way.

Field in the synchrotron pulse

The final result

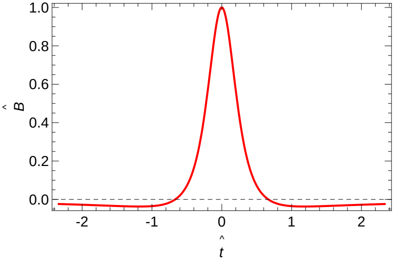
$$B_y = \frac{Z_0 q}{\pi r \rho} \frac{\gamma^{-2} - \xi^2}{(\xi^2 + \gamma^{-2})^3}, \qquad t = \frac{r}{c} + \frac{\rho}{c}\left(\frac{1}{2\gamma^2}\xi + \frac{1}{6}\xi^3\right). \quad (20.1)$$

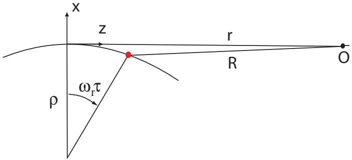
The variable $\xi = c\tau/\rho = \omega_r \tau/\beta$ has a simple physical meaning—it is roughly the angle on the orbit from which radiation that arrives at the observation point O originates [remember our original concept of waves emitted by an accelerated particle]. Polarization: the electric field of radiation is in the plane of the orbit.
Introduce the dimensionless time variable $\hat{t} = (\gamma^3 c/\rho)(t - r/c)$ and the dimensionless magnetic field $\hat{B} = (\pi r \rho / Z_0 q \gamma^4) B_y$:

$$\hat{B} = \frac{1 - \zeta^2}{(\zeta^2 + 1)^3}, \qquad \hat{t} = \frac{1}{2}\zeta + \frac{1}{6}\zeta^3, \quad (20.2)$$

where $\zeta = \xi\gamma$.

Time profile of the pulse





The characteristic width of the pulse $\Delta\hat{t} \sim 1$, which means that the duration of the pulse in physical units

$$\Delta t \sim \frac{\rho}{c\gamma^3}\,. \tag{20.3}$$

The spectrum of frequencies presented in the radiation is $\Delta\omega \sim c\gamma^3/\rho$.

Fourier transformation of the radiation field and the radiated power

The power $\mathcal{P}$ radiated in unit solid angle $d\Omega$ in the x-z plane is, using the plane wave approximation,

$$\frac{d\mathcal{P}}{d\Omega} = r^2 \mathbf{S} \cdot \mathbf{n} = \frac{r^2 c}{\mu_0} B_y^2(t) \tag{20.4}$$

The total energy flux $\mathcal{W}$ in this plane is

$$\frac{d\mathcal{W}}{d\Omega} = r^2 \int_{-\infty}^{\infty} dt \; S(t) \; .$$

We consider the *spectrum* of the radiation. Use Parseval's theorem:

$$\int_{-\infty}^{\infty} dt \; B_y(t)^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega |\tilde{B}_y(\omega)|^2 \quad = \frac{1}{\pi} \int_0^{\infty} d\omega |\tilde{B}_y(\omega)|^2 \, ,$$

where

$$\tilde{B}_y(\omega) = \int_{-\infty}^{\infty} dt \; B_y(t) e^{i\omega t} \, .$$

Fourier transformation of the radiation field

We introduce the energy radiated per unit frequency interval per unit solid angle as

$$\frac{d^2\mathcal{W}}{d\omega d\Omega} = \frac{r^2 c^2}{\pi Z_0} |\tilde{B}_y(\omega)|^2 \,, \tag{20.5}$$

so that the total energy radiated per unit solid angle is

$$\frac{d\mathcal{W}}{d\Omega} = \int_0^\infty d\omega \frac{d^2\mathcal{W}}{d\omega d\Omega} \,. \tag{20.6}$$

[$\frac{d\mathcal{W}}{d\Omega}$ and $\frac{d^2\mathcal{W}}{d\omega d\Omega}$ are not derivatives of the function $\mathcal{W}$, just a notation.]

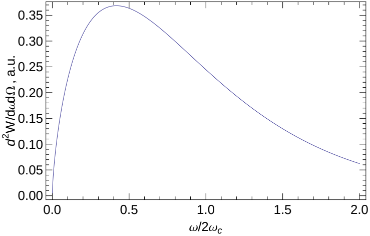
Fourier transformation of the radiation field

The function $\tilde{B}_y(\omega)$ is calculated in the Lecture notes. The spectrum is

$$\frac{d^2\mathcal{W}}{d\omega d\Omega} = \frac{q^2 Z_0}{12\pi^3}\left(\frac{\rho\omega}{c}\right)^2\left(\frac{1}{\gamma^2}\right)^2 K_{2/3}^2\left(\frac{\omega}{2\omega_c}\right), \qquad (20.7)$$

where $K_{2/3}$ is the MacDonald function, and the critical frequency

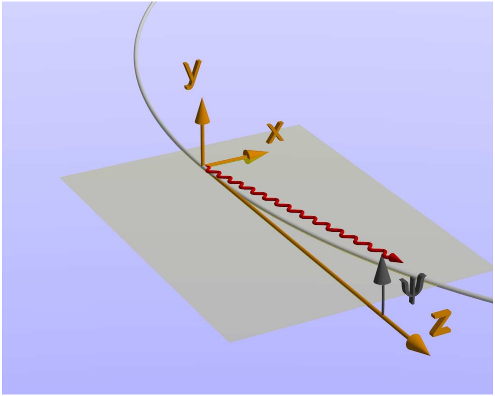
$$\omega_c = \frac{3c\gamma^3}{2\rho}. \qquad (20.8)$$



The dominant part of the spectrum is in the region $\omega \sim \omega_c$. For NSLS-II $\omega \sim \omega_c$ corresponds to the wavelength of 0.5 nm or 2.4 keV photon energy.

Synchrotron radiation for $\psi \neq 0$

In a more general case of radiation at an angle $\psi \neq 0$ the calculation is more involved. We will summarize some of the results of this general case.



Synchrotron radiation for $\psi \neq 0$

A more general formula valid for $\psi \neq 0$ is

$$\frac{d^2\mathcal{W}}{d\omega d\Omega} = \frac{q^2 Z_0}{12\pi^3}\left(\frac{\rho\omega}{c}\right)^2\left(\frac{1}{\gamma^2}+\psi^2\right)^2\left[K_{2/3}^2(\chi) + \frac{\psi^2}{1/\gamma^2+\psi^2}K_{1/3}^2(\chi)\right], \tag{20.9}$$

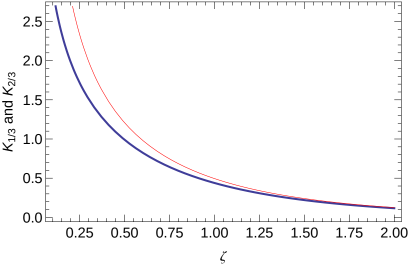
where

$$\chi = \frac{\omega\rho}{3c}\left(\frac{1}{\gamma^2}+\psi^2\right)^{3/2} = \frac{\omega}{2\omega_c}\left(1+\psi^2\gamma^2\right)^{3/2}. \tag{20.10}$$

This result was obtained by J. Schwinger in 1949. Setting $\psi = 0$ we recover Eq. (20.7).

Synchrotron radiation for $\psi \neq 0$

The Bessel functions falls off when the argument $\chi \gg 1$.

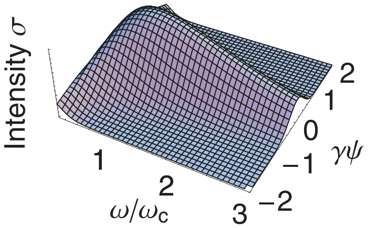


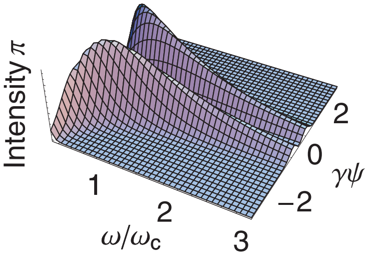
The spectrum is strongly correlated with the angle. What is the angular spread in ψ? The dominant part of the radiation is in the region $\chi \lesssim 1$. If $\omega \sim \omega_c$, then $\psi \lesssim 1/\gamma$. For lower frequencies, the angle is larger:

$$\psi \sim \frac{1}{\gamma}\left(\frac{\omega_c}{\omega}\right)^{1/3} \sim \left(\frac{\lambda\!\!\!^{-}}{\rho}\right)^{1/3} \tag{20.11}$$

Synchrotron radiation for $\psi \neq 0$

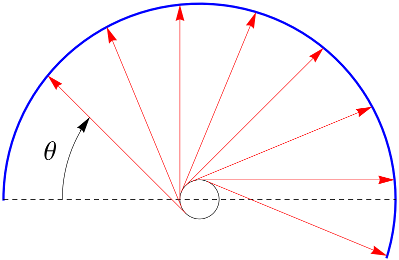
The two terms in the square brackets correspond to different polarizations of the radiation. The first one is the so called σ-mode, it has polarization with E_x and B_y. The second one has the polarization with the electric field E_y and the magnetic field B_x; it is called the π mode.





Synchrotron radiation for $\psi \neq 0$

The radiation is localized at small angles ψ.



when integrating over $d\Omega$, in addition to integration over the angle ψ, one should include integration over the angle θ,

$$\frac{d\mathcal{W}}{d\omega} = \int d\Omega \frac{d^2\mathcal{W}}{d\omega d\Omega} = \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} \cos\psi d\psi \frac{d^2\mathcal{W}}{d\omega d\Omega}$$

$$\approx 2\pi \int_{-\infty}^{\infty} d\psi \frac{d^2\mathcal{W}}{d\omega d\Omega}$$

Synchrotron radiation for $\psi \neq 0$

Integration over the angle gives the frequency distribution

$$\frac{d\mathcal{W}}{d\omega} = \frac{2\pi\rho}{c}\frac{q^2\gamma Z_0 c}{9\pi\rho} S\left(\frac{\omega}{\omega_c}\right) , \qquad (20.12)$$

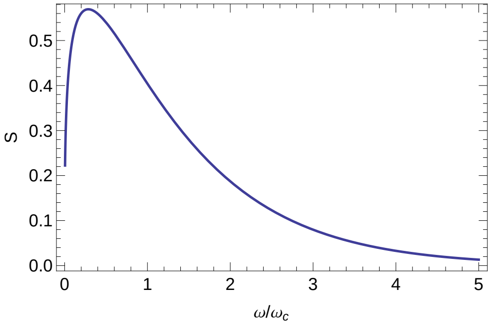
where

$$\begin{aligned} S(x) =& \frac{27x^2}{16\pi^2}\int_{-\infty}^{\infty} d\tau \left(1+\tau^2\right)^2 \\ & \cdot \left[K_{2/3}^2\left(\frac{x}{2}(1+\tau^2)^{3/2}\right) + \frac{\tau^2}{1+\tau^2}K_{1/3}^2\left(\frac{x}{2}(1+\tau^2)^{3/2}\right)\right] \\ =& \frac{9\sqrt{3}}{8\pi}x\int_x^{\infty} K_{5/3}(y)\, dy \ . \end{aligned}$$

The last expression is not easy to derive, but it is the most common definition used.

Synchrotron radiation for $\psi \neq 0$

Plot of function S.



The function S is normalized to one: $\int_0^\infty dx S(x) = 1$.

Synchrotron radiation for $\psi \neq 0$

For small and large values of the argument we have the asymptotic expressions

$$
\begin{aligned}
S(x) &= \frac{27}{8\pi}\frac{\sqrt{3}}{2^{1/3}}\Gamma\left(\frac{5}{3}\right)x^{1/3}\,, && x \ll 1 \\
S &= \frac{9}{8}\sqrt{\frac{3}{2\pi}}\sqrt{x}e^{-x}\,, && x \gg 1\,.
\end{aligned}
\tag{20.13}
$$

Total radiated power

Integrating $d\mathcal{W}/d\omega$ over all frequencies, we will find the total energy W_r radiated in one revolution

$$W_r = \int_0^\infty d\omega \frac{d\mathcal{W}}{d\omega} = \frac{2\pi\rho}{c} \cdot \frac{q^2 \gamma Z_0 c}{9\pi\rho} \omega_c \,. \qquad (20.14)$$

The radiation power (energy radiation per unit time) by a single electron is

$$\mathcal{P} = \frac{W_r}{2\pi\rho/c} = \frac{Z_0 c q^2 \gamma}{9\pi\rho} \omega_c = \frac{2 r_0 m c^2 \gamma^4 c}{3\rho^2} \,. \qquad (20.15)$$

Note that $\mathcal{P}/c$ is the energy radiated by one electron per unit length of path.

Number of photons per unit bandwidth per unit time is:

$$\frac{d^2 N_{ph}}{dt\, d\omega} = \frac{1}{\hbar\omega} \frac{d\mathcal{W}}{d\omega} \frac{1}{2\pi\rho/c} = \frac{8}{27} \alpha \frac{1}{\gamma^2} \frac{\omega_c}{\omega} S\left(\frac{\omega}{\omega_c}\right) \,. \qquad (20.16)$$

Total radiated power

Let's calculate the power of synchrotron radiation from dipole magnets in NSLS-II. The energy 3 GeV, bending magnetic field 0.4 T, current $I = 400$ mA. Bending radius $\rho = p/eB = 25$ m, $C = 780$ m.

The critical frequency $\omega_c = 3.6 \times 10^{18}$ 1/s corresponding to the wavelength $\lambda = 0.5$ nm and the photon energy 2.4 keV.

Radiated power by one electron 8.8×10^{-8} W [note that an electron radiates this power only inside the dipole and dipoles occupy a small fraction of the ring]. Number of electrons in the ring $N = (C/c)I/e = 8.1 \times 10^{12}$ which the gives total power of 0.14 MW. [There is additional radiation due to the wigglers in the ring].

Synchrotron radiation reaction force (Lecture 24)

February 3, 2016

Lecture outline

In contrast to the earlier section on synchrotron radiation, we compute the fields observed close to the beam. We can use this to evaluate the forces electrons are subject to from the fields produced by the rest of the beam. Two different methods of calculation will be shown. The second one is rather strange but it works.
We end by computing the synchrotron radiation reaction force of a relativistic particle and show, by explicit calculations for a Gaussian bunch, that the work of this force is equal to the energy radiated per unit time.

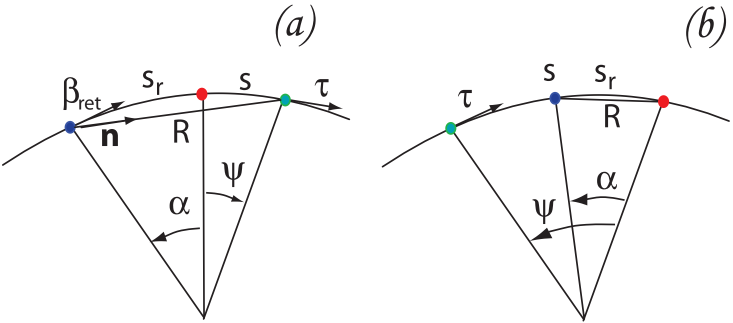
CSR wake

When a bunch of charged particle emits radiation, the energy of the electromagnetic field is taken from its kinetic energy. The energy balance in the process is maintained through a force that acts in the direction opposite to the velocity of the bunch. This force is called the *radiation reaction force*. In this lecture we consider a different situation where the effect of coherent synchrotron radiation fields fulfils the energy balance.

To simplify calculations, we will systematically neglect terms of the order of $1/\gamma$ in our derivation. This means that we consider the limit $\gamma \to \infty$ and $\beta = 1$. An additional advantage of this approach is that we automatically neglect the longitudinal Coulomb field of the bunch, that is proportional to γ^{-2}.

We also assume a thin bunch, $\sigma_\perp \to 0$. Our goal is to find the tangential component of the electric field inside the bunch. This is called the *CSR wake*.

CSR wake close to beam



Geometry for (a) $\psi > 0$, observer in front of particle, and (b) $\psi < 0$, observer behind particle. Green = observer, red = particle at same time t, blue = particle at time t_{ret}.
Observer is path length s in front of particle at time t. Negative sign for location behind particle. Angle $\psi = s/\rho$, ρ is bending radius.

CSR wake close to beam

Define α similarly, particle has moved path length $\alpha\rho$ from location at retarded time. Always positive. Velocity $\approx c$, $\gamma \to \infty$.

Distance R from retarded position to observer:
$R = 2\rho \sin(s/2\rho) < s$. For straight paths, we expected head of bunch not to feel any forces from tail; long propagation distance until fields catch up. But in bend, path of particle is *not* the shortest distance between 2 points.

Retarded time:

$$\begin{aligned} t - t_{\mathrm{ret}} =& \frac{\rho\alpha}{c} \\ =& \frac{R}{c} = \frac{2\rho}{c}\left|\sin\left(\frac{\alpha+\psi}{2}\right)\right| \ . \end{aligned} \tag{24.1}$$

CSR wake close to beam

Parameter s is defined to follow the beam. $s = S - ct$, S in fixed co-ordinates. We use the following:

$$\begin{aligned}
\mathbf{A} &= \boldsymbol{\beta}_{\text{ret}} \Phi / c \ , \\
A_s &= \mathbf{A} \cdot \boldsymbol{\tau} = (\boldsymbol{\beta}_{\text{ret}} \cdot \boldsymbol{\tau}) \Phi / c \ , \\
\Phi &= \Phi(S - ct) \ , \qquad \mathbf{A} = \mathbf{A}(S - ct) \ .
\end{aligned}$$

Then

$$E_s = -\frac{\partial \Phi}{\partial S} - \frac{\partial A_s}{\partial t} = -\frac{\partial}{\partial s}(\Phi - cA_s) = -\frac{\partial}{\partial s}[\Phi(1 - \boldsymbol{\beta}_{\text{ret}} \cdot \boldsymbol{\tau})] \ . \tag{24.2}$$

Note that

$$\begin{aligned}
\boldsymbol{\beta}_{\text{ret}} \cdot \boldsymbol{\tau} &\simeq \cos(\alpha + \psi) \\
\boldsymbol{\beta}_{\text{ret}} \cdot \mathbf{n} &\simeq \cos\left[\frac{\pi}{2} - \frac{1}{2}(\pi - \alpha - \psi)\right] = \cos\left(\frac{\alpha + \psi}{2}\right) \ .
\end{aligned} \tag{24.3}$$

CSR wake close to beam

Positive ψ:

$$\Phi - cA_s = \frac{q}{4\pi\epsilon_0} \frac{1 - \boldsymbol{\beta}_{\rm ret} \cdot \boldsymbol{\tau}}{R(1 - \boldsymbol{\beta}_{\rm ret} \cdot \mathbf{n})}$$
$$= \frac{q}{4\pi\epsilon_0} \frac{1 - \cos(\alpha + \psi)}{1 - \cos[(\alpha + \psi)/2]} \simeq \frac{q}{4\pi\epsilon_0} \frac{4}{\rho\alpha} \; . \tag{24.4}$$

From

$$\alpha = 2 \sin\left(\frac{\alpha + \psi}{2}\right) \simeq \alpha + \psi - \frac{1}{24}(\alpha + \psi)^3 \; . \tag{24.5}$$

find $\alpha \simeq (24\psi)^{1/3} \ll \psi$ for small $\psi \ll 1$.
Then longitudinal field is

$$E_s = -\frac{\partial}{\partial s}\left(\Phi - cA_s\right) = \frac{q}{4\pi\epsilon_0} \frac{2}{3^{4/3}\rho^{2/3}s^{4/3}} \tag{24.6}$$

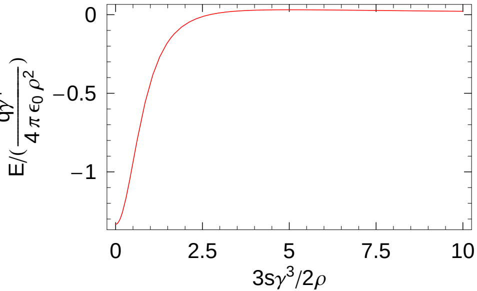
For negative ψ, fields are negligible.

CSR wake close to beam

There is a strong singularity near $s = 0^+$. This is not physical, when $s \sim \rho/\gamma^3$ the fields level off and actually switch sign. $E_{s,\max} \sim q\gamma^4/(4\pi\epsilon_0\rho^2)$.

We will wind up restricting attention to $\omega \ll \omega_c$. For bunch lengths much longer than $c/\omega_c \sim \rho/\gamma^3$ the approximate expression should be accurate enough.

More accurate CSR result



A more exact calculation of the single-particle CSR wake, taking into account scale finite γ and lengths of order ρ/γ^3. This function is needed to show full energy conservation; the other expression is adequate for looking at frequencies $\ll \omega_c$, which is all we need for bunches with $\sigma_z \gg c/\omega_c$.

Gaussian current distribution

For lone particles, the radiation reaction force has to be determined by conservation of energy, namely that the particle has to lose an amount of energy equal to the energy emitted in the radiation fields. For a whole bunch, at least the low-frequency part of the radiated spectrum is balanced by collective fields from the whole bunch. The collective fields scale as N^2 rather than as N for incoherent radiation.

For the coherent radiation, we need to take into account the current profile. Assume a Gaussian distribution, $\mu(s) = Nq(2\pi)^{-1/2}\sigma_z^{-1}e^{-s^2/2\sigma_z^2}$. We also define $\lambda = \mu/Nq$.

Gaussian current distribution

The mean or collective longitudinal field is given by

$$
\begin{aligned}
\mathcal{E}_s(s) =& N \int_{-\infty}^{\infty} E_s(s-s')\lambda(s')\, ds' \\
=& - N \int_{-\infty}^{s} (\Phi - cA_s)|s-s' \frac{d\lambda(s')}{ds'} \\
=& - \frac{Nq}{4\pi\epsilon_0} \frac{2}{3^{1/3}\rho^{2/3}} \int_{-\infty}^{s} \frac{1}{(s-s')^{1/3}} \frac{d\lambda(s')}{ds'} \ ,
\end{aligned}
\tag{24.7}
$$

where we integrated by parts.

Gaussian current distribution

The last integral can be computed numerically, and is shown below. The inaccurate singularity has minimal impact on the results.

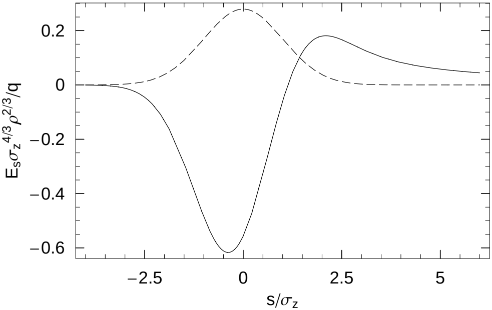


Figure : CSR field of a Gaussian bunch. The distance is measured in units of σ_z, and the field is measured in units of $Q/\sigma_z^{4/3}\rho^{2/3}$, where Q is the total charge of the bunch.

Assumptions behind the CSR wake calculation

We started the CSR analysis by examining fields close to the beam. This should be the near field, but we took the limit $\gamma \to \infty$. The fields are ultimately derived from the Liénard-Wiechert potential,

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0} \frac{\mathbf{n} - \boldsymbol{\beta}_{\rm ret}}{\gamma^2 R^2 (1 - \boldsymbol{\beta}_{\rm ret} \cdot \mathbf{n})^3} + \frac{q}{4\pi\epsilon_0 c} \frac{\mathbf{n} \times \{(\mathbf{n} - \boldsymbol{\beta}_{\rm ret}) \times \dot{\boldsymbol{\beta}}_{\rm ret}\}}{R(1 - \boldsymbol{\beta}_{\rm ret} \cdot \mathbf{n})^3},$$
$$\mathbf{B} = \mathbf{n} \times \mathbf{E}. \tag{24.8}$$

It seems like we should ignore the first term when considering radiation. Also, the second term dominates when $\omega > \omega_c$. We are in fact neglecting the second term with $\dot{\boldsymbol{\beta}}_{\rm ret}$. We do this not because of the single particle radiation field, but because we assume that the bunch current only varies on a scale length $\gg c/\omega_c$. This effectively suppresses radiation at the higher frequencies, and for frequencies much lower than ω_c, the $\dot{\boldsymbol{\beta}}_{\rm ret}$ is a small correction.

Coherent versus incoherent synchrotron radiation

If we are ignoring acceleration, then we can treat any electron as moving in a straight line, so long as that line crosses the appropriate location at t_{ret}.

There will be some radiation at high frequencies, but this will come from fluctuations in the local density and the radiated power scales like N, the total number of particles in the bunch, rather than with N^2. This corresponds to incoherent synchrotron radiation, not coherent radiation. Our approximations are valid for the coherent part.

Radiation reaction field in a bunch of particles

As we pointed out at the beginning of the lecture, the longitudinal field keeps the energy balance between the kinetic energy of the particle and the radiation. Let us demonstrate by direct calculation for a Gaussian bunch,

$$\lambda(s) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma_z} e^{-s^2/2\sigma_z^2} \ , \tag{24.9}$$

that this is indeed the case. First we calculate the energy W_r that the beam loses in one turn around the ring

$$\begin{aligned} W_r = & - Nqc \frac{2\pi\rho}{c} \int_{-\infty}^{\infty} ds\, \mathcal{E}_s(s)\lambda(s) \qquad (24.10) \\ & = \frac{N^2 q^2 \rho^{1/3}}{3^{1/3}\epsilon_0} \int_{-\infty}^{\infty} \lambda(s) ds \int_{-\infty}^{s} \frac{1}{(s-s')^{1/3}} \frac{\partial \lambda(s')}{\partial s'}\, ds' \\ & = \frac{N^2 q^2 \rho^{1/3}}{3^{1/3}\epsilon_0\, 2\pi\, \sigma_z^{4/3}} 2\pi \int_{-\infty}^{\infty} \hat{\lambda}(x) dx \int_{-\infty}^{x} \frac{1}{(x-y)^{1/3}} \frac{\partial \hat{\lambda}(y)}{\partial y}\, dy \ , \end{aligned}$$

where $\hat{\lambda}$ takes s/σ_z as an argument.

Radiation reaction field in a bunch of particles

We then have to compare this expression with the power of coherent synchrotron radiation. The latter is calculated through the expression for coherent radiation (see Lecture 23)

$$\left.\frac{d\mathcal{W}}{d\omega}\right|_{\text{bunch}} = \frac{d\mathcal{W}}{d\omega}\left(N + N^2 F(\omega)\right) \tag{24.11}$$

where the intensity $d\mathcal{W}/d\omega$ is taken in the limit of low frequencies as given in Lecture 20:

$$\frac{d\mathcal{W}}{d\omega} = \frac{2\pi\rho}{c}\frac{q^2\gamma Z_0 c}{9\pi\rho} S\left(\frac{\omega}{\omega_c}\right) , \tag{24.12}$$

$$S(x) \simeq \frac{27}{8\pi}\frac{\sqrt{3}}{2^{1/3}}\Gamma\left(\frac{5}{3}\right) x^{1/3}, \qquad x \ll 1 .$$

Coherent radiation

The form factor

$$F(\omega) = \left| \int_{-\infty}^{\infty} ds \lambda(s) e^{i\omega s/c} \right|^2 , \qquad (24.13)$$

The form factor is equal the square of the absolute value of the Fourier transform of the longitudinal distribution function of the beam.
For a Gaussian distribution function

$$\lambda(s) = \frac{1}{\sqrt{2\pi}\sigma_z} e^{-s^2/2\sigma_z^2} , \qquad (24.14)$$

we have

$$F(\omega) = e^{-(\omega\sigma_z/c)^2} . \qquad (24.15)$$

Comparison

This gives the radiated energy per turn as

$$N^2 \int_{-\infty}^{\infty} d\omega F(\omega) \frac{d\mathcal{W}}{d\omega} = \frac{2}{9} q^2 \gamma Z_0 N^2 \int_{-\infty}^{\infty} d\omega F(\omega) S\left(\frac{\omega}{\omega_c}\right)$$
$$= N^2 q^2 \gamma Z_0 \frac{3}{4\pi {\omega_c}^{1/3}} \frac{\sqrt{3}}{2^{1/3}} \Gamma\left(\frac{5}{3}\right) \int_{-\infty}^{\infty} d\omega e^{-\omega^2 \sigma_z^2/c^2} \omega^{1/3}$$
$$= \frac{N^2 q^2 \rho^{1/3}}{3^{1/3} \epsilon_0 \; 2\pi \sigma_z^{4/3}} \frac{3\sqrt{3}}{2} \Gamma\left(\frac{5}{3}\right) \int_{-\infty}^{\infty} d\omega e^{-\omega^2} \omega^{1/3} . \qquad (24.16)$$

The easiest way to verify that this is consistent with the energy decrease of the electron beam given in Eq. 24.10 is to compare the values of the purely numerical terms following the first fraction. And, indeed, calculations give that they are both equal to 3.17594966. (The above integral over frequency yields $\Gamma(2/3)$).

Formation length of radiation and coherence (Lecture 23)

February 4, 2016

Lecture outline

It takes some volume of free space for a particle to generate radiation. We estimate the longitudinal and transverse size of this volume for the synchrotron radiation. We then analyze the radiation of a bunch of particles.

Formation length

What length of the trajectory is involved in the formation of the radiation? In Eq. (20.1) the variable $\xi = c\tau/\rho$ is related to the retarded time τ. We saw that the characteristic width of the electromagnetic pulse in terms of ξ is $\Delta\xi \sim \gamma^{-1}$, for a time duration $\tau \sim \rho/c\gamma$. Hence the length of the orbit necessary for formation of the radiation pulse, which we call the *formation length*, l_f is

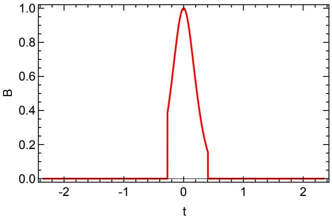
$$l_f \sim c\tau \sim \frac{\rho}{\gamma} \,. \tag{23.1}$$

One does not need a complete circular orbit to generate synchrotron radiation. If the length of the magnet $L_m \gg \rho/\gamma$, all our results are valid. NSLS-II magnets: $\rho = 25$ m, $\gamma = 6 \times 10^3$, $l_f \approx 15$ cm.

Longitudinal formation length

How does this formation length agree with the duration of the radiation pulse of the order of $\rho/c\gamma^3$? Since the charge is moving with the velocity $v \approx c(1 - 1/2\gamma^2)$, the relative velocity between the charge and the electromagnetic field is $\Delta v \sim c/\gamma^2$, and during the formation time τ the field propagates away from the charge at the distance $\Delta v \tau \sim \rho/c\gamma^3$, which is the duration of the pulse.

For a short magnet, $L_m \lesssim \rho/\gamma$, only a fraction of the pulse will be radiated.



Jumps in the pulse profile generate the *edge* radiation.

Longitudinal formation length

More precisely, what we found above is the formation length for the bulk of radiation with the characteristic frequency $\omega \sim \omega_c$.

What is the formation length for frequency $\omega \ll \omega_c$? The analysis requires a more careful look into integrals involved in the derivation of the spectrum from which it follows that

$$l_f(\omega) \sim \rho^{2/3} \lambda\!\!\!^{-}{}^{1/3}, \tag{23.2}$$

for $\lambda\!\!\!^{-} \lesssim \lambda\!\!\!^{-}_c$. For the critical frequency $\omega = \omega_c$ this formula gives us the previous expression.
In quantum language, the formation length gives time for a virtual photon carried by the electromagnetic field of a particle to free from the charge and become a real photon.

Transverse coherence length

A charge also needs some space in the direction perpendicular to the orbit to release radiation. We can evaluate *transverse coherence length* $l_\perp$:

$$l_\perp \sim l_f \Delta\psi \sim \rho^{1/3}\lambda^{2/3} \,. \tag{23.3}$$

(we used (20.11)).

The practical importance of the transverse coherence is that the radiation can be suppressed by metal walls, if they are put close to the beam. More specifically, if the beam propagates through a dipole magnet in a metal pipe of radius a, then the radiation with wavelength $\lambda \gtrsim \sqrt{a^3/\rho}$ is suppressed. This is called a *shielding* effect and it is important for suppression of undesirable coherent radiation of short bunches in accelerators.

Shielding of coherent radiation

The suppression factor for synchrotron radiation when a particle is moving (on a circular orbit) between two parallel perfectly conducting plates in the plane equally removed from each plate. The distance between the plates is $2h$.

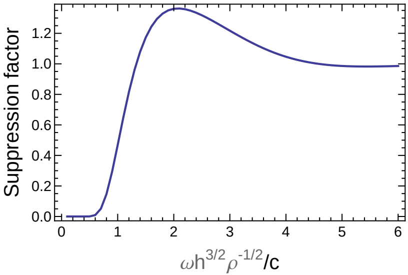


Figure : Suppression factor for the intensity of the synchrotron radiation for the case of parallel conducting plates as a function of frequency.

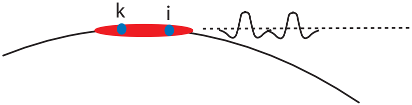
Transverse coherence length

The result shown above is valid in the limit $\omega \ll \omega_c$. Note that the horizontal axis in the plot is $\omega h^{3/2}\rho^{-1/2}/c \sim (h/l_\perp)^{3/2}$, and one can see that the suppression factor approaches zero when h becomes much smaller than $l_\perp$.

The concepts of the longitudinal and transverse formation lengths is very general and is applicable to any kind of radiation processes.

Coherent radiation

Consider radiation of a bunch of particles. Neglect the transverse size of the bunch and take into account the longitudinal distribution, $\lambda(s)$, $\int \lambda(s) ds = 1$.



The Fourier transform of a single pulse magnetic field $B(t)$ of a reference particle that passes through the center of the coordinate system at $t = 0$ is

$$\tilde{B}(\omega) = \int_{-\infty}^{\infty} dt B(t) e^{i\omega t} . \tag{23.4}$$

The field radiated by the bunch is sum of pulses

$$\mathcal{B}(t) = \sum_{i=1}^{N} B(t - t_i) , \tag{23.5}$$

where $t_i = s_i/c$ with s_i the position of the particle i in the bunch.

Coherent radiation

The Fourier image of this field is

$$\tilde{\mathcal{B}}(\omega) = \int dt \mathcal{B} e^{i\omega t} = \sum_{i=1}^{N} \int dt B(t - t_i) e^{i\omega t} = \sum_{i=1}^{N} \tilde{B}(\omega) e^{i\omega t_i} \,.$$

The spectral intensity of the radiation is proportional to $|\tilde{\mathcal{B}}(\omega)|^2$ (see Eq. (20.5))

$$\begin{aligned}
|\tilde{\mathcal{B}}(\omega)|^2 &= \left| \sum_{i=1}^{N} \tilde{B}(\omega) e^{i\omega t_i} \right|^2 = |\tilde{B}(\omega)|^2 \sum_{i,k} e^{i\omega(t_i - t_k)} \\
&= |\tilde{B}(\omega)|^2 \left(N + \sum_{i \neq k} e^{i\omega(t_i - t_k)} \right) \\
&= N |\tilde{B}(\omega)|^2 + 2 |\tilde{B}(\omega)|^2 \sum_{i<k} \cos\left(\omega \frac{s_i - s_k}{c} \right) \,. \qquad (23.6)
\end{aligned}$$

Coherent radiation

The first term in the last equation is *incoherent* radiation—it is proportional to the number of particles in the beam. The second one is the *coherent* radiation term. The number of terms in the last sum is $N(N-1)/2 \approx N^2/2$. Instead of doing summation we can average $\cos(\omega(s_i - s_k)/c)$ assuming that s_i and s_k are distributed with the probability given by $\lambda(s)$:

$$
\begin{aligned}
2\sum_{i<k} \cos\left(\omega \frac{s_i - s_k}{c}\right) &\approx N^2 \int ds' ds'' \lambda(s')\lambda(s'') \cos\left(\omega \frac{s' - s''}{c}\right) \\
&= N^2 F(\omega)\,,
\end{aligned}
\tag{23.7}
$$

where the *form factor* $F(\omega)$ is

$$
F(\omega) = \int ds' ds'' \lambda(s')\lambda(s'') \cos\left(\omega \frac{s' - s''}{c}\right)\,, \tag{23.8}
$$

and

$$
\left.\frac{d\mathcal{W}}{d\omega}\right|_{\text{bunch}} = \frac{d\mathcal{W}}{d\omega}(N + N^2 F(\omega))\,. \tag{23.9}
$$

Coherent radiation

Eq. (23.8) can also be written as

$$F(\omega) = \left| \int_{-\infty}^{\infty} ds \lambda(s) e^{i\omega s/c} \right|^2 , \qquad (23.10)$$

Eq. (23.10) shows that the form factor is equal the square of the absolute value of the Fourier transform of the longitudinal distribution function of the beam.
For the Gaussian distribution function

$$\lambda(s) = \frac{1}{\sqrt{2\pi}\sigma_z} e^{-s^2/2\sigma_z^2} , \qquad (23.11)$$

we have

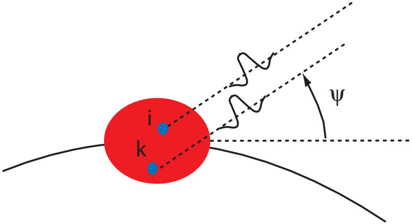
$$F(\omega) = e^{-(\omega\sigma_z/c)^2} . \qquad (23.12)$$

Coherent radiation

For $\omega/c \gg 1/\sigma_z$ the form factor is exponentially small. For the reduced wavelengths longer than the bunch length, $ƛ \gtrsim \sigma_z$, the power scales as the number of particles squared. This radiation by a factor of N is larger than the *incoherent* radiation. For a bunch with $N \sim 10^{10}$ this makes a huge difference! However, this radiation can only occur at long wavelengths, and those are in many cases (but not always) shielded by walls.

Effect of the transverse size of the beam

We considered the above radiation in the longitudinal direction. We now take into account the radiation at an angle and consider a 3D distribution of the beam. The 3D distribution function is $\lambda(r)$ normalizes so that $\int d^3 r \lambda(r) = 1$.



Effect of the transverse size of the beam

We see that the delay between pulses radiated by the central particle and a particle located at position r in the bunch is equal to $\Delta t = (r_i - r_k) \cdot n/c$. The field (23.6) can now be written as

$$|\tilde{\mathcal{B}}(\omega)|^2 = N|\tilde{B}(\omega)|^2 + 2|\tilde{B}(\omega)|^2 \sum_{i<k} \cos\left(\omega \frac{n \cdot (r_i - r_k)}{c}\right) , \qquad (23.13)$$

which gives for the form factor

$$F(\omega) = \int d^3r' d^3r'' \lambda(r')\lambda(r'') \cos\left(\omega \frac{n \cdot (r' - r'')}{c}\right) . \qquad (23.14)$$

Similar to transition from (23.8) to (23.10) one can show that (23.14) can be written as

$$F(\omega, n) = \left| \int d^3r \lambda(r) e^{i\omega n \cdot r/c} \right|^2 , \qquad (23.15)$$

that is the square of the absolute value of the three dimensional Fourier transform of the distribution function.

Coherent synchrotron radiation

Calculate the coherent synchrotron radiation using Eqs. (23.9). The dominant contribution to the coherent radiation comes from frequencies of the order or $\omega \sim \sigma_z/c$ which are much smaller than ω_c. Hence for the intensity $d\mathcal{W}/d\omega$ we can use the low frequencies approximation given by Eqs. (20.12) and (20.13) and assume a Gaussian distribution function

$$
\begin{aligned}
N^2 \int_{-\infty}^{\infty} d\omega F(\omega) \frac{d\mathcal{W}}{d\omega} &= \frac{2}{9} q^2 \gamma Z_0 N^2 \int_{-\infty}^{\infty} d\omega F(\omega) S\left(\frac{\omega}{\omega_c}\right) \\
&= N^2 q^2 \gamma Z_0 \frac{3}{4\pi {\omega_c}^{1/3}} \frac{\sqrt{3}}{2^{1/3}} \Gamma\left(\frac{5}{3}\right) \int_{-\infty}^{\infty} d\omega e^{-\omega^2 \sigma_z^2/c^2} \omega^{1/3} \\
&= 0.2 \frac{N^2 q^2 \gamma Z_0}{{\omega_c}^{1/3}} \left(\frac{c}{\sigma_z}\right)^{4/3} . \qquad (23.16)
\end{aligned}
$$

Coherent synchrotron radiation

NSLS-II: Number of bunches 1040, bunch length $\sigma_z = 2.9$ mm. This would give the total radiation power of 87 kW. This power is actually shielded by the conducting wall of the vacuum chamber.

Coherent synchrotron radiation

From R. Carr et al., Nature, 2002.

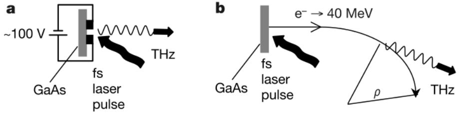


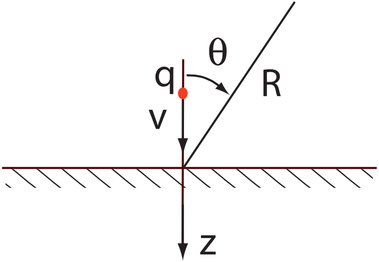
Figure 1 Comparison between coherent THz radiation generated by an 80-MHz conventional laser-driven THz source (**a**) and the relativistic source described here (**b**). In **a**, the photo-induced carriers immediately experience a force from the bias field ($\sim$100 V across a 100 μm gap) of $\sim 10^6\,\mathrm{V\,m^{-1}}$, which results in an acceleration of $10^{17}\,\mathrm{m\,s^{-2}}$. The entire process is completed in less than 1 ps, resulting in spectral content up to a few THz. In **b**, approximately the same number of charge carriers are brought to a relativistic energy of $>$10 MeV in a linac, after which a magnetic field bends their path into a circle of radius $\rho = 1$ m, resulting in an acceleration $c^2/\rho = 10^{17}\,\mathrm{m\,s^{-2}}$, the same as for **a**. An observer of **b** would also detect a brief pulse of electromagnetic radiation as an electron bunch passed by. But in this case, two factors control the pulse duration; one is the bunch length, and the other is the time for the relativistically compressed acceleration field from each electron to sweep past. The latter is given approximately by[28] $\delta t = 4\rho/(3\gamma^3 c)$, and determines the spectral range emitted by each electron. The bunch length determines the spectral range over which the coherent enhancement occurs. For an electron energy of 10 MeV ($\gamma = 21$), and with $\rho = 1$ m, we obtain a δt of about 500 fs, which is comparable to the bunch length. The resulting spectral content extends up to about 1 THz, the same spectral range as for **a**. With all factors except γ the same, we see from equation (1) that the power radiated by a relativistic electron exceeds that from a conventional THz emitter by a factor of $\gamma^4 = 21^4 = 2\times10^5$.

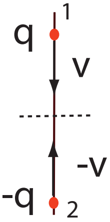
Transition and diffraction radiation (Lecture 22)

February 4, 2016

Transition radiation

The transition radiation (TR) occurs when a charge traveling with a constant velocity crosses a boundary that separates two media with different electric properties. We will calculate the TR for the case when a charge hits a plane metal surface, moving in the direction perpendicular to the surface.





Transition radiation

We replace the metal with an image charge. The boundary conditions in this case are satisfied automatically. The charges collide at point O at time $t = 0$ and annihilate. At time $t > 0$ there are no charges in the system. [Method of images]

The trajectories of particles 1 and 2 for $t < 0$ are given by $r_1(t) = (0, 0, vt)$ and $r_2(t) = (0, 0, -vt)$ respectively. We also need to define the retarded times for both particles, $t_{\mathrm{ret}}^{(1)}$ and $t_{\mathrm{ret}}^{(2)}$. They satisfy equations $c(t - t_{\mathrm{ret}}^{(1)}) = |R - r_1(t_{\mathrm{ret}}^{(1)})|$ and $c(t - t_{\mathrm{ret}}^{(2)}) = |R - r_2(t_{\mathrm{ret}}^{(2)})|$ correspondingly (see (18.1)), again for $t_{\mathrm{ret}}^{(1)} < 0$ and $t_{\mathrm{ret}}^{(2)} < 0$. Note that the moment $t_{\mathrm{ret}}^{(1)} = t_{\mathrm{ret}}^{(2)} = 0$ corresponds to $r_1 = r_2 = 0$ and $t = R/c$; we will use this observation below.

Transition radiation

To calculate the radiation, we need to find the vector potential A at the observation point. For $t_{\rm ret}^{(1)} < 0$ and $t_{\rm ret}^{(2)} < 0$ this is the potential corresponding to two charges, and for $t_{\rm ret}^{(1)} > 0$ and $t_{\rm ret}^{(2)} > 0$, when there are no charges in the system, $A = 0$. As noted above $t_{\rm ret} = 0$ corresponds to $t = R/c$, hence, for $t < R/c$ we can use Eq. (18.5)

$$A = \frac{Z_0}{4\pi}\left(\boldsymbol{\beta}\frac{q}{R_1(t_{\rm ret}^{(1)})(1-\boldsymbol{\beta}\cdot n)} + (-\boldsymbol{\beta})\frac{(-q)}{R_2(t_{\rm ret}^{(2)})(1+\boldsymbol{\beta}\cdot n)}\right) \times h\left(\frac{R}{c} - t\right)$$

where $\boldsymbol{\beta} = (0, 0, v/c)$, h is the step function, and

$$R_1(t) = \sqrt{(z-vt)^2 + x^2 + y^2}, \qquad R_2(t) = \sqrt{(z+vt)^2 + x^2 + y^2}.$$

Note: R here is the same as r earlier.

Transition radiation

The magnetic field of the radiation is given by

$$B = -\frac{1}{c} n \times \frac{\partial A}{\partial t} \,. \tag{22.1}$$

When we differentiate the equation for A with respect to time, we only need to differentiate the function h—differentiating R_1 and R_2 would give a field that decays faster than $1/R$. The result is

$$B = \frac{Z_0}{4\pi} \frac{q}{c} \delta\left(\frac{R}{c} - t\right) \left(\frac{1}{R_1(0)(1+\beta\cos\theta)} + \frac{1}{R_2(0)(1-\beta\cos\theta)}\right) \cdot (n \times \boldsymbol{\beta}) \tag{22.2}$$

where n is a unit vector in the direction of R. [Polarization]

Transition radiation

The values of R_1 and R_2 in this equation should be taken at the retarded time $t_{\mathrm{ret}} = 0$:

$$R_1 = R_2 = \sqrt{z^2 + x^2 + y^2} = R\,, \tag{22.3}$$

and

$$B = \frac{Z_0}{4\pi}\frac{2q}{Rc}\delta\left(\frac{R}{c} - t\right)\frac{n \times \boldsymbol{\beta}}{1 - \beta^2 \cos^2\theta}\,. \tag{22.4}$$

We see that the radiation field is an infinitely thin spherical wave propagating from the point of entrance to the metal. Since the Fourier transform of the delta function is a constant, we conclude that the spectrum of the radiation does not depend on the frequency.

Transition radiation

The spectrum of the radiation is given by Eq. (20.5) with

$$\tilde{B}(\omega) = \int_{-\infty}^{\infty} dt B(t) e^{i\omega t} = \frac{Z_0}{4\pi} \frac{2qe^{i\omega R/c}}{Rc} \frac{n \times \boldsymbol{\beta}}{1 - \beta^2 \cos^2\theta} \,. \qquad (22.5)$$

For the angular distribution of the spectral power we have

$$\frac{d^2\mathcal{W}}{d\omega d\Omega} = \frac{c^2 R^2}{\pi Z_0} |\tilde{B}(\omega)|^2 = \frac{Z_0 q^2}{4\pi^3} \frac{\beta^2 \sin^2\theta}{(1 - \beta^2 \cos^2\theta)^2} \,. \qquad (22.6)$$

It follows from this equation that for a relativistic particle the dominant part of the radiation goes in the backward direction. Using $\beta^2 = 1 - \gamma^{-2}$ and approximating $\sin\theta \approx \theta$ and $\cos^2\theta \approx 1 - \theta^2$ we find

$$\frac{d^2\mathcal{W}}{d\omega d\Omega} \approx \frac{Z_0 q^2}{4\pi^3} \frac{\theta^2}{(\gamma^{-2} + \theta^2)^2} \,. \qquad (22.7)$$

Transition radiation

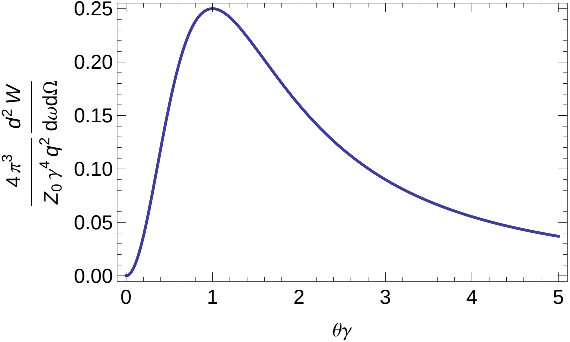


Figure : Angular distribution of transition radiation for a relativistic particle. [Note: vertical axis should be normalized to γ^2, not γ^4.]

Transition radiation

One can integrate this equation to find the spectrum of the transition radiation

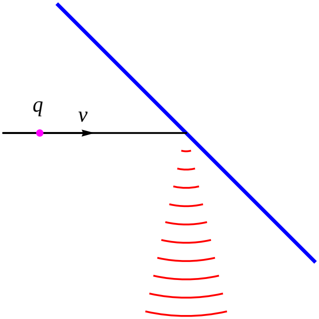
$$\frac{d\mathcal{W}}{d\omega} = 2\pi \int_{\pi/2}^{\pi} \sin\theta d\theta \frac{d^2\mathcal{W}}{d\omega d\Omega}$$
$$= \frac{Z_0 q^2}{4\pi^2} \left[\left(\frac{1}{\beta} + \beta \right) \operatorname{arctanh}(\beta) - 1 \right] . \tag{22.8}$$

[can not replace $\sin\theta$ by θ].

The spectrum of the radiation does not depend on the frequency. Formally, integrating over ω from zero to infinity, we will find that the total radiated energy diverges. In reality, the spectrum is cut off at high frequencies because metals lose their capability of being perfect conductors, and the transition radiation is suppressed. This occurs at $\hbar\omega \sim 10 - 30$ eV (submicron wavelength).

Transition radiation

Problem 22.2. *The usual setup in the experiment for the optical transition radiation (OTR) diagnostic is shown in figure below: the beam passes through a metal foil tilted at the angle 45 degrees relative to the beam orbit. Show that in this case the radiation propagates predominantly in the direction perpendicular to the orbit. How to solve this problem using the method of image charges?*



Transition radiation

From R. Fiorito and D. Rule. OPTICAL TRANSITION RADIATION BEAM EMITTANCE DIAGNOSTICS.

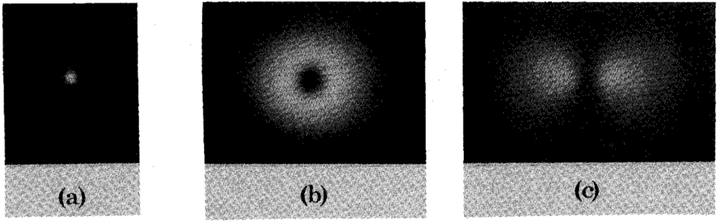


Fig. 2 a) Beam image in OTR, b) unpolarized, c) horizontally polarized OTR angular distributions.

Transition radiation in the forward direction

United States Patent [19] [11] **Patent Number: 4,763,344**

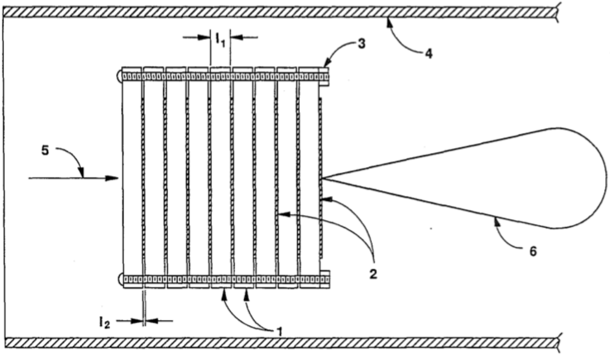
Piestrup [45] **Date of Patent: Aug. 9, 1988**

[54] **X-RAY SOURCE FROM TRANSITION RADIATION USING HIGH DENSITY FOILS**

[76] Inventor: **Melvin A. Piestrup**, 13800 Skyline Blvd., Woodside, Calif. 94062

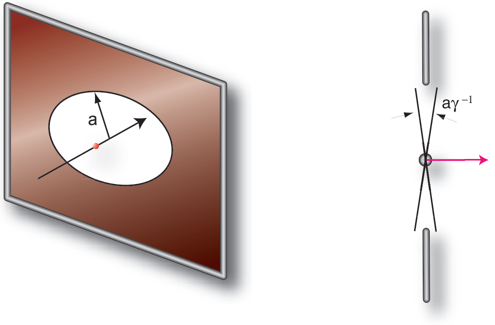
trons in Periodic Radiators," Phys. Rev. D vol. 10, pp. 3594–3607, Dec. 1974.

A. N. Chu, M. A. Piestrup, T. W. Barbee, Jr., and R. H. Pantell, "Transition Radiation as a Source of X-rays," J. Appl. Phys vol. 51, pp. 1290–1293, Mar. 1980.



Diffraction radiation

Interception of the beam with a foil either destroys it or deteriorates the beam properties. Sometimes one would like to generate radiation without strongly perturbing the beam. This can be achieved if the beam passes through a hole in a metal foil—and generates so called *diffraction radiation*. The radiation properties depend on the size and the shape of the hole. The complete electromagnetic solution of the radiation problem in this case requires methods which are beyond the scope of this course.



a) b)

Diffraction radiation

It can be shown that in the limit $\gamma \gg 1$ and $\theta \ll 1$ the angular spectral distribution of the diffraction radiation is given by the following formula

$$\frac{d^2\mathcal{W}}{d\omega d\Omega} \approx \frac{Z_0 q^2}{4\pi^3} \frac{\theta^2}{(\gamma^{-2}+\theta^2)^2} F\left(\frac{\omega a\theta}{c}, \frac{\omega a}{c\gamma}\right), \tag{22.9}$$

where

$$F(x,y) = \left(yJ_2(x)K_1(y) - \frac{y^2}{x}J_1(x)K_2(y)\right)^2, \tag{22.10}$$

with $J_{1,2}$ the Bessel functions and $K_{1,2}$ the modified Bessel functions.

Diffraction radiation

In the limit $a \to 0$ the function $F \to 1$ and we recover the result of the transition radiation (22.7). The hole has a small effect on the transition radiation at a given frequency ω if it is small, $a \ll c\gamma/\omega$. We plot the spectral intensity of the radiation as a function of the angle θ for several values of the parameter $a\omega/c\gamma$.

Diffraction radiation

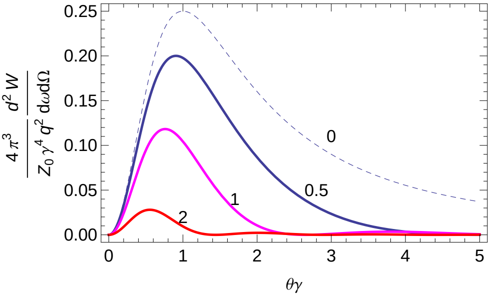


Figure : Angular distribution of the diffraction radiation for various values of the parameter $a\omega/c\gamma$ (indicated by numbers near the curves). The dashed line shows the limit $a \to 0$, corresponding to the case of the transition radiation.

Diffraction radiation